NATIONAL EXAMINATIONS MAY 2013

04-BS-5 ADVANCED MATHEMATICS

3 Hours duration

NOTES:

1. If doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper a clear statement of any assumption made.

2. Candidates may use one of the approved Casio or Sharp calculators. This is a Closed Book Exam. However, candidates are permitted to bring **ONE** aid sheet (8.5"x11") written on both sides.

3. Any five (5) questions constitute a complete paper. Only the first five answers as they appear in your answer book will be marked.

4. All questions are of equal value.

Marking Scheme

1. 20 marks

2. (a) 14 marks ; (b) 6 marks

3. (a) 5 marks ; (b) 9 marks ; (c) 3 marks ; (d) 3 marks

4. (A) 10 marks ; (B) 10 marks

5. 20 marks

6. (A) 10 marks ; (B) 10 marks

7. (a) 6 marks ; (b) 6 marks ; (c) 8 marks

1.Consider the following differential equation

$$(x+4)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+2y=0$$

Find two linearly independent solutions about the ordinary point x=0.

2.(a) Find the Fourier series expansion of the periodic function f(x) of period p=2π.

$$f(x) = 2\pi x - x^2 \qquad 0 \le x \le 2\pi$$

(b) Use the result obtained in (a) to prove that $\frac{\pi^2}{12}=\sum_{n=1}^{\infty}\frac{(-1)^{(n+1)}}{n^2}$

3.Consider the following function where a is a positive constant

$$f(x) = \begin{cases} 2a\cos(ax) & -\frac{\pi}{2a} < x < \frac{\pi}{2a} \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute the area bounded by f(x) and the x-axis. Graph f(x) against x for $a=1.0$.and $a=2.0$

(b) Find the Fourier transform F(ω) of f(x).

(c) Graph F(ω) against ω for the same two values of a mentioned in (a). Explain what happens to f(x) and F(ω) when a tends to infinity.

Note: $$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\exp(-i\omega x)dx$$

4.(A) Prove that the coefficients α and β of the least-squares parabola $y = \alpha x + \beta x^2$ that fits the set of n points (x_i, y_i) can be obtained as follows:

$$\alpha = \frac{(\sum_{i=1}^{i=n} x_i^4)(\sum_{i=1}^{i=n} x_i y_i) - (\sum_{i=1}^{i=n} x_i^3)(\sum_{i=1}^{i=n} x_i^2 y_i)}{(\sum_{i=1}^{i=n} x_i^2)(\sum_{i=1}^{i=n} x_i^4) - (\sum_{i=1}^{i=n} x_i^3)^2}; \quad \beta = \frac{(\sum_{i=1}^{i=n} x_i^2)(\sum_{i=1}^{i=n} x_i^2 y_i) - (\sum_{i=1}^{i=n} x_i^3)(\sum_{i=1}^{i=n} x_i y_i)}{(\sum_{i=1}^{i=n} x_i^2)(\sum_{i=1}^{i=n} x_i^4) - (\sum_{i=1}^{i=n} x_i^3)^2}$$

4(B) The following table displays the exact values of the quadratic function f(x) at the four indicated values of the independent variable x. Obtain the second-degree Lagrange polynomial that fits the first three points and then use the fourth point to check the correctness of your answer.

x	−2	1	2	4
F(x)	11	−4	3	35

5.The following results were obtained in a certain experiment:

x	−1.00	−0.75	−0.50	−0.25	0	0.25	0.50	0.75	1.00
y	3.50	12.50	14.00	17.50	16.00	13.50	12.00	12.50	18.50

Use Romberg's algorithm to obtain an approximation of the area bounded by the unknown curve represented by the table and the lines x = −1, x = 1 and y = 0.

Note: The Romberg algorithm produces a triangular array of numbers, all of which are numerical estimates of the definite integral $\int_a^b f(x)dx$. The array is denoted by the following notation:

R(1,1)			
R(2,1)	R(2,2)		
R(3,1)	R(3,2)	R(3,3)	
R(4,1)	R(4,2)	R(4,3)	R(4,4)

where

$$R(1,1) = \frac{H_1}{2}[f(a) + f(b)]$$

$$R(k,1) = \frac{1}{2}\left[R(k-1,1) + H_{k-1}\sum_{n=1}^{n=2^{k-2}} f(a + (2n-1)H_k)\right]; \qquad H_k = \frac{b-a}{2^{k-1}}$$

$$R(k,j) = R(k,j-1) + \frac{R(k,j-1) - R(k-1,j-1)}{4^{j-1} - 1}$$

6.(A) The equation $x^5 - 16x^2 - 20 = 0$ has a root close to $x_0 = 2.6$. Use the following iterative formula twice to find a better approximation of this root. (Note: Carry seven digits in your calculations)

$$x_{i+1} = x_i - \frac{f(x_i)}{f^{(1)}(x_i) - \dfrac{f(x_i)f^{(2)}(x_i)}{2f^{(1)}(x_i)}}$$

Hint:Let $f(x) = x^5 - 16x^2 - 20$. Note that $f^{(1)}(x)$ and $f^{(2)}(x)$ denote the first and second *derivative of* $f(x)$ *respectively.*

6.(B) The equation $\ln(x + 2) - x^2 + 7 = 0$ has a root in the neighbourhood of $x_0 = 2.95$. Write the equation in the form $x = g(x)$ and use the method of fixed-point iteration five times to find a better approximation of this root. (Note: Carry seven digits in your calculations)

7.(a) Consider the matrices $A = \begin{bmatrix} 5 & -1 & -2 \\ -1 & 3 & 4 \\ -2 & 4 & 6 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and

$O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Prove that $A^3 - 14A^2 + 42A - 8U = O$.

(b) The equation given in (a) can be rewritten as follows

$$A^{-1} = \frac{1}{8}(A^2 - 14A + 42U)$$

Use this last equation to find the inverse A^{-1} of A.

(c) Use the inverse A^{-1} obtained in (b) to solve the following system of three linear equations:

$$\begin{aligned} 5x_1 - x_2 - 2x_3 &= 9 \\ -x_1 + 3x_2 + 4x_3 &= -6 \\ -2x_1 + 4x_2 + 6x_3 &= -11 \end{aligned}$$