National Exams May 2018

16-Elec-B1, Digital Signal Processing

3 hours duration

NOTES:

1. If doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper, a clear statement of any assumptions made.

2. This is a Closed Book exam.
   Candidates may use one of two calculators, the Casio or Sharp approved models. They are also entitled to one aid sheet with tables & formulas written both sides. No textbook excerpts or examples solved.

3. FIVE (5) questions constitute a complete exam.
   Clearly indicate your choice of any five of the six questions given otherwise the first five answers found will be considered your pick.

4. All questions are worth 12 points.
   See below for a detailed breakdown of the marking.

Marking Scheme

1. (a) 4, (b) 4, (c) 4, total = 12
2. (a) 5, (b) 1, (c) 1, (d) 2, (e) 3, total = 12
3. (a) 6, (b) 6, total = 12
4. (a) 8, (b) 4, total = 12
5. (a) 4, (b) 4, (c) 4, total = 12
6. (a) 3, (b) 9, total = 12

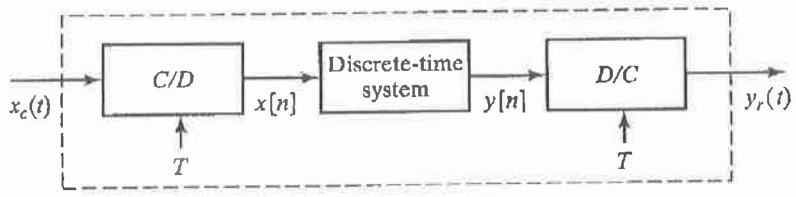
The number beside each part above indicates the points that part is worth

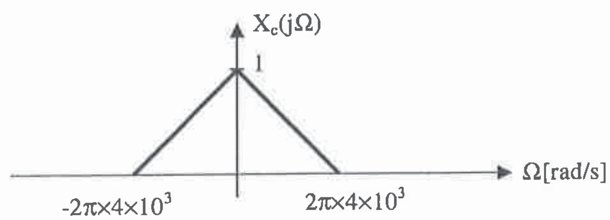
1.- The discrete-time system in the figure below is an ideal lowpass filter with cutoff frequency of $\pi/6$ rad/s.

(a) If $x_c(t)$ is bandlimited to 4kHz as indicated in the figure, determine the maximum value of $\boldsymbol{T}$ that will avoid aliasing in the C/D converter.

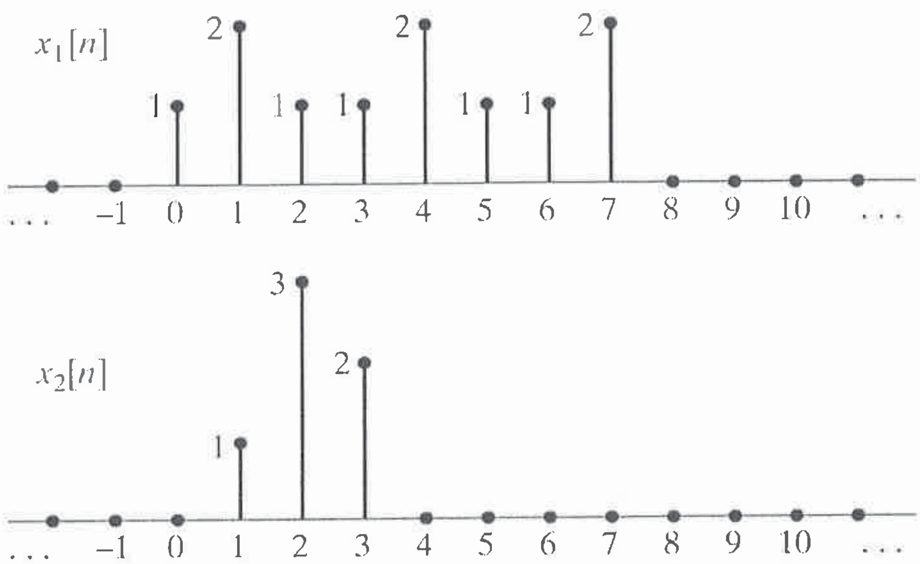
(b) If $\mathbf{1/T}$ = 24 kHz, what is the cutoff frequency of the effective continuous-time filter?

(c) Sketch and label the Fourier transform of $y_r(t)$ using the sampling rate in part (b) and assuming the passband gain of the ideal lowpass discrete-time filter is 1.





2.- Two finite-length signals, $x_1[n]$ and $x_2[n]$, are sketched in the figure below. Assume that $x_1[n]$ and $x_2[n]$ are zero outside of the interval shown in the figure. Let $x_3[n]$ be the eight-point circular convolution of $x_1[n]$ with $x_2[n]$; i.e., $x_3[n] = x_1[n] \circledast_8 x_2[n]$.



(a) Determine $x_3[n]$ using the circular convolution theorem.

Let $x_4[n]$ be the linear convolution of $x_1[n]$ with $x_2[n]$.

(b) What is the value of n for the first non-zero sample of $x_4[n]$?

(c) What is the value of n for the last non-zero sample of $x_4[n]$?

(d) Find and sketch $x_4[n]$.

(e) Use $x_4[n]$ to find $x_3[n]$ and verify the result obtained in part (a).

3.- Consider a causal LTI system with impulse response $h[n]$ and system function

$$H(z) = \frac{(1 - 2z^{-1})(1 - 4z^{-1})}{z\left(1 - \frac{1}{2}z^{-1}\right)}$$

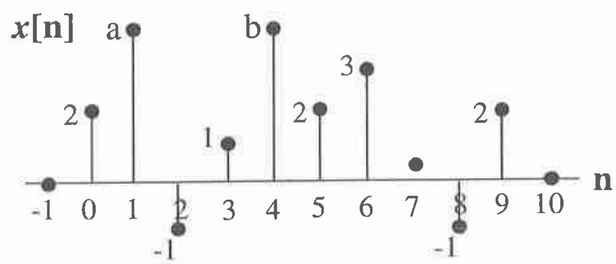
(a) Draw a direct form II flow graph for the system.

(b) Draw the transposed form of the flow graph in part (a).

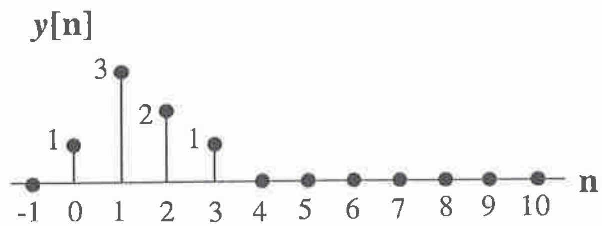
4.- The figure below shows a ten-point discrete-time sequence $x[n]$. Assume that $x[n] = 0$ outside the interval shown. The values of $x[1]$ and $x[4]$ are not known and represented by a and b. Note that these two values are not necessarily drawn to scale in the figure.

Let $X(e^{j\omega})$ be the DTFT of $x[n]$ and $Y[k]$ be samples of $X(e^{j\omega})$ every $\pi/2$; *i.e.*,

$$Y[k] = X(e^{j\omega})\big|_{\omega=(\pi/2)k}, \quad 0 \le k \le 3.$$



(a) The sequence $y[n]$ is the 4-point inverse DFT of $Y[k]$. Check if the four-point sequence shown in the figure below could be $y[n]$. Justify.



(b) Is it possible to find values for a and b that satisfy $y[n]$ having some or all of the values shown in the figure? If so, give the values needed for a and b, otherwise explain.

5.- Consider an LTI system with input $x[n]$ and output $y[n]$ that satisfies the difference equation

$$y[n] - \frac{5}{2}y[n-1] + y[n-2] = x[n] - x[n-1].$$

(a) Find the system function $H(z)$.

(b) Identify all possible regions of convergence (ROC) for $H(z)$.

(c) Determine the value of the impulse response at $n = 0$, $h[0]$, for each of the possibilities listed in part (b).

6.- The design of an FIR filter using the windowing method is based on a desired frequency response $H_d(e^{j\omega})$ and its corresponding desired impulse response $h_d[n]$. More specifically, the resulting filter impulse response $h[n]$ is found as:

$$h[n] = h_d[n] \cdot w[n] ,$$

where $w[n]$ is a time window, *e.g.* Hamming, Hanning, etc, non-zero for n = 0, 1, 2, … , M.

(a) Indicate which of the following linear-phase FIR filter types can be based on each of the desired frequency responses given below (write 'YES' in the corresponding square).

(b) Below each 'YES' justify your answer for that square based on the symmetry of the resulting $h[n]$ and whether that type supports the frequency selective filter sought by $H_d(e^{j\omega})$.

$H_d(e^{j\omega})$	Type I	Type II	Type III	Type IV
$\begin{cases} e^{-j\frac{M}{2}\omega}, & \lvert\omega\rvert < \omega_c \\ 0, & \omega_c < \lvert\omega\rvert < \pi \end{cases}$				
$\begin{cases} 0, & \lvert\omega\rvert < \omega_c \\ e^{-j\frac{M}{2}\omega}, & \omega_c < \lvert\omega\rvert < \pi \end{cases}$				
$\begin{cases} -je^{-j\frac{M}{2}\omega}, & -\pi < \omega < -\omega_c \\ 0, & \lvert\omega\rvert < \omega_c \\ je^{-j\frac{M}{2}\omega}, & \omega_c < \omega < \pi \end{cases}$				
$\begin{cases} -je^{-j\frac{M}{2}\omega}, & -\omega_{c2} < \omega < -\omega_{c1} \\ je^{-j\frac{M}{2}\omega}, & \omega_{c1} < \omega < \omega_{c2} \\ 0, & elsewhere \end{cases}$				

Additional Information

(Not all of this information is necessarily required today!)

DTFT Synthesis Equation	DTFT Analysis Equation
$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$	$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$
Parseval's Theorem	**N-point DFT**
$E = \sum_{n=-\infty}^{\infty}\|x[n]\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\|X(e^{j\omega})\|^2 d\omega$	$X[k] = \sum_{n=0}^{N-1} x[n]\,W_N^{kn}, \quad W_N = e^{-j\frac{2\pi}{N}}$
Z-transform of a sequence $x[n]$	**Sinusoidal response of LTI systems, real h[n]**
$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$	$y[n] = \|H(e^{j\omega_0})\| \cos\left(\omega_0 n + \measuredangle H(e^{j\omega_0})\right)$

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\|R_x$
4	3.4.4	$nx[n]$	$-z\frac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Properties of the Discrete Fourier Transform

Finite-Length Sequence (Length N)	N-point DFT (Length N)
1. $x[n]$	$X[k]$
2. $x_1[n], x_2[n]$	$X_1[k], X_2[k]$
3. $ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
4. $X[n]$	$Nx[((-k))_N]$
5. $x[((n-m))_N]$	$W_N^{km} X[k]$
6. $W_N^{-\ell n} x[n]$	$X[((k-\ell))_N]$
7. $\sum_{m=0}^{N-1} x_1(m) x_2[((n-m))_N]$	$X_1[k]X_2[k]$
8. $x_1[n]x_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1} X_1(\ell)X_2[((k-\ell))_N]$
9. $x^*[n]$	$X^*[((-k))_N]$
10. $x^*[((-n))_N]$	$X^*[k]$
11. $\mathcal{R}e\{x[n]\}$	$X_{\text{ep}}[k] = \frac{1}{2}\{X[((k))_N] + X^*[((-k))_N]\}$
12. $j\mathcal{J}m\{x[n]\}$	$X_{\text{op}}[k] = \frac{1}{2}\{X[((k))_N] - X^*[((-k))_N]\}$
13. $x_{\text{ep}}[n] = \frac{1}{2}\{x[n] + x^*[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$
14. $x_{\text{op}}[n] = \frac{1}{2}\{x[n] - x^*[((-n))_N]\}$	$j\mathcal{J}m\{X[k]\}$
Properties 15–17 apply only when $x[n]$ is real.	
15. Symmetry properties	$\begin{cases} X[k] = X^*[((-k))_N] \\ \mathcal{R}e\{X[k]\} = \mathcal{R}e\{X[((-k))_N]\} \\ \mathcal{J}m\{X[k]\} = -\mathcal{J}m\{X[((-k))_N]\} \\ \lvert X[k]\rvert = \lvert X[((-k))_N]\rvert \\ \sphericalangle\{X[k]\} = -\sphericalangle\{X[((-k))_N]\} \end{cases}$
16. $x_{\text{ep}}[n] = \frac{1}{2}\{x[n] + x[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$
17. $x_{\text{op}}[n] = \frac{1}{2}\{x[n] - x[((-n))_N]\}$	$j\mathcal{J}m\{X[k]\}$

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
9. $[\cos\omega_0 n]u[n]$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
10. $[\sin\omega_0 n]u[n]$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\|z\| > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\|z\| > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$

Initial Value Theorem:

If $x[n]$ is a causal sequence, *i.e.* $x[n] = 0,\ \forall n < 0$, then

$$x[0] = \lim_{z\to\infty} X(z)$$