NATIONAL EXAMINATIONS DECEMBER 2017

04-BS-5 ADVANCED MATHEMATICS

3 Hours duration

NOTES:

1. If doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper a clear statement of any assumption made.

2. Candidates may use one of the approved Casio or Sharp calculators. This is a Closed Book Exam. However, candidates are permitted to bring **ONE** aid sheet (8.5"x11") written on both sides.

3. Any five (5) questions constitute a complete paper. Only the first five answers as they appear in your answer book will be marked.

4. All questions are of equal value.

Marking Scheme

1. 20 marks

2. (A) 14 marks ; (B) 6 marks

3. (a) 5 marks ; (b) 9 marks ; (c) 3 marks ; (d) 3 marks

4. (A) 14 marks ; (B) 6 marks

5. 20 marks

6. (a) 7 marks ; (b) 7 marks ; (c) 6 marks

7. (a) 10 marks ; (b) 10 marks

1.Find the eigenvalues and eigenfunctions of the following Sturm-Liouville problem:

$$\frac{d}{dx}(x^{-5}\frac{dy}{dx}) + (9+\lambda)x^{-7}y = 0 \quad ; \quad y(1) = 0 \quad ; \quad y(e^2) = 0$$

2.(A) Find the Fourier series expansion of the periodic function $f(x)$ of period $p = 2\pi$. Make a neat sketch of $f(x)$ against x.

$$f(x) = \begin{cases} x(\pi + x) & -\pi < x < 0 \\ x(\pi - x) & 0 \le x < \pi \end{cases}$$

2.(B) Use the result obtained in (A) to prove the following

$$\frac{\pi^3}{32} = \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)}}{(2n-1)^3}$$

3.Consider the following function where a is a positive constant:

$$f(x) = \begin{cases} \frac{2}{a}\exp(x/a) & x \le 0 \\ \frac{2}{a}\exp(-x/a) & x > 0 \end{cases}$$

(a)Compute the area bounded by $f(x)$ and the x-axis. Graph $f(x)$ against x for $a = 1$ and $a = 0.5$.

(b)Find the Fourier transform $F(\omega)$ of $f(x)$..

(c)Graph $F(\omega)$ against ω for the same two values of a mentioned in (a).

(d)Explain what happens to $f(x)$ and $F(\omega)$ when a tends to zero.

Note: $$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\exp(-i\omega x)dx$$

4.(A) Use the method of Lagrange to find the polynomial of highest possible degree that fits the following set of five points.

x	-3	-1	0	1	4
$f(x)$	0	-36	0	16	-56

4(B).Use the forward difference formulas supplied below to find the approximate value of the first, second, third and fourth derivative of the function $f(x)$ tabulated below at $x=-3$. (Hint: Let $x_0=-3$ and $h=1$)

x	−4	−3	−2	−1	0	1	2	3	4	5
$f(x)$	600	233	86	39	20	5	18	131	464	1,185

$$f'(x_0)\approx\frac{1}{12h}[-25f(x_0)+48f(x_0+h)-36f(x_0+2h)+16f(x_0+3h)-3f(x_0+4h)]$$

$$f''(x_0)\approx\frac{1}{12h^2}[35f(x_0)-104f(x_0+h)+114f(x_0+2h)-56f(x_0+3h)+11f(x_0+4h)]$$

$$f'''(x_0)\approx\frac{1}{2h^3}[-5f(x_0)+18f(x_0+h)-24f(x_0+2h)+14f(x_0+3h)-3f(x_0+4h)]$$

$$f''''(x_0)\approx\frac{1}{h^4}[f(x_0)-4f(x_0+h)+6f(x_0+2h)-4f(x_0+3h)+f(x_0+4h)]$$

5. The following results were obtained in a certain experiment:

x	1	1.50	2	2.50	3	3.50	4	4.50	5
y	215	345	444	537	600	763	856	955	1,085

Use Romberg's algorithm to obtain an approximation of the area bounded by the unknown curve represented by the table and the lines $x=1$, $x=5$ and the x-axis.

Note: The Romberg algorithm produces a triangular array of numbers, all of which are numerical estimates of the definite integral $\int_a^b f(x)dx$. The array is denoted by the following notation:

R(1,1)
R(2,1) R(2,2)
R(3,1) R(3,2) R(3,3)
R(4,1) R(4,2) R(4,3) R(4,4)

where

$$R(1,1)=\frac{H_1}{2}[f(a)+f(b)] \quad ;$$

$$R(k,1)=\frac{1}{2}\left[R(k-1,1)+H_{k-1}\sum_{n=1}^{n=2^{k-2}}f(a+(2n-1)H_k)\right] \quad ; \quad H_k=\frac{b-a}{2^{k-1}}$$

$$R(k,j)=R(k,j-1)+\frac{R(k,j-1)-R(k-1,j-1)}{4^{j-1}-1}$$

6.(a) The equation $x^4 - 3x^3 + 5x - 6 = 0$ can be written in the form $x = g(x)$ in several ways. Find the form which will enable you to find the root that lies in the neighbourhood of $x_0 = 2$. Perform six iterations. (Note: Carry seven digits in your calculations).

(b) The equation given in (a) has another root between $a = -2$ and $b = -1$. Apply the method of bisection three times to find a better approximation of this root. Then use Newton's method three times to find a better approximation of this root. (Note: Carry seven digits in your calculations)

(c) Find the derivative $f'(x)$ of the function $f(x) = x^4 - 3x^3 + 5x - 6$. One of the roots of $f'(x) = 0$ is equal to one. Find the other two roots. Use this information to find any maximum or minimum that the function $f(x)$ may have. Use the graph paper provided to sketch as neatly as possible the graph of $f(x)$ against x.

7.The matrix A $= \begin{bmatrix} 3 & -9 & 9 \\ -4 & 17 & 8 \\ 1 & -5 & 1 \end{bmatrix}$ can be written as the product of a lower triangular matrix L$= \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$ and an upper triangular matrix

U $= \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$, that is A=LU.

(a) Find L and U.

(b) Use the results obtained in (a) to solve the following system of three linear equations:

$$3x - 9y + 9z = -24$$
$$-4x + 17y + 8z = 7$$
$$x - 5y + z = -7$$