National Exams May 2015

07-Elec-B1, Digital Signal Processing

3 hours duration

NOTES:

1. If doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper, a clear statement of any assumptions made.

2. This is a Closed Book exam.
   Candidates may use one of two calculators, the Casio or Sharp approved models. They are also entitled to one aid sheet with tables & formulas written both sides.

3. FIVE (5) questions constitute a complete exam.
   Clearly indicate your choice of any five of the six questions given otherwise the first five answers found will be considered your pick.

4. All questions are worth 12 points.
   See below for a detailed breakdown of the marking.

Marking Scheme

1. total = 12
2. (a) 4, (b) 4, (c) 4, total = 12
3. (a) 6, (b) 6, total = 12
4. (a) 3, (b) 3, (c) 3, (d) 3, total = 12
5. (a) 4, (b) 2, (c) 3, (d) 3, total = 12
6. (a) 4, (b) 4, (c) 4, total = 12

The number beside each part above indicates the points that part is worth

1.- Let $x[n]$ be a purely real discrete-time signal. You are given the following information about $x[n]$ and you are asked to find and sketch $x[n]$. Even if you are unable to specify $x[n]$ fully, you may receive partial credit by describing which features or values of $x[n]$ are determined by each clue.

(a) $x[n]$ is a causal sequence.

(b) Let $v[n] = x[n+2]$. The discrete-time Fourier transform $V(e^{j\omega})$ is purely real.

(c) $\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|X(e^{j\omega})\right|^2 d\omega = 24.$

(d) $\lim_{z\to\infty} X(z) = 3.$

(e) $\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})e^{j\omega}d\omega = 1.$

(f) $x[2] < 0$.

Note: Consult tables and formulas provided at the end of this exam as needed

2.- Suppose $x_c(t)$ is a periodic continuous-time signal with period 1 ms and for which the Fourier series is

$$x_c(t) = \sum_{k=-9}^{9} a_k e^{j(2\pi kt/10^{-3})}$$

The Fourier series coefficients a_k are zero for $|k| > 9$.

$x_c(t)$ is sampled with a sample spacing $T = \frac{1}{6}\times 10^{-3}$ s to form $x[n]$. That is,

$$x[n] = x_c\left(n\cdot\frac{1}{6}10^{-3}\right).$$

(a) Is $x[n]$ periodic and, if so, with what period? [4 pts]

(b) Is the sampling rate above the Nyquist rate? That is, is T sufficiently small to avoid aliasing? Explain. [4 pts]

(c) Find the Fourier series coefficients of $x[n]$ in term of a_k. [4 pts]

3.- A causal and stable LTI system has impulse response $h[n]$. The pole-zero plot for the system function $H(z)$ is shown in the figure below. There are three finite poles at $z = 1/2$ & $\pm j/2$ and two finite zeros at $z = 0$ & -1.

For each of the transformations undergone by the original impulse response $h[n]$ in parts (a) & (b) below:

i. Sketch the pole-zero plot of the resulting system functions; $H_1(z)$ & $H_2(z)$.

ii. Justify or show how the new pole and zero locations are obtained. [3 pts each]

Note: Clearly label your axes and the rectangular coordinates of each zero/pole.

iii. Determine the region of convergence (ROC) of the resulting system functions. [1 pt each]

Based on the ROC's and pole-zero plots obtained indicate if the new systems are

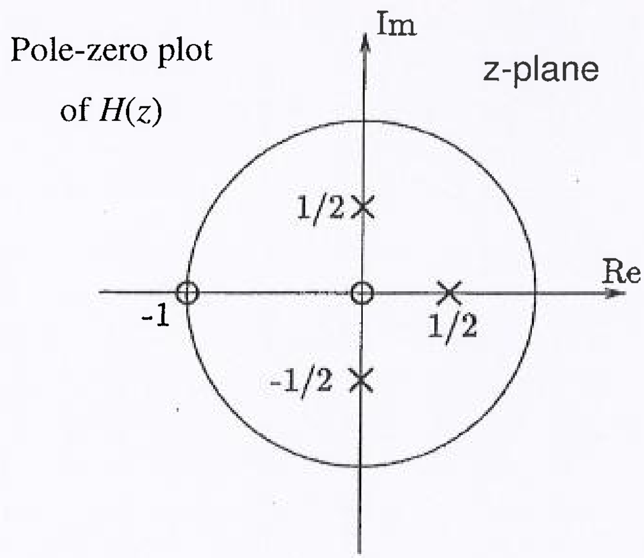
iv. Causal, and/or [1 pt each]

v. Stable. [1 pt each]

Justify causality and stability based on the pole-zero plots obtained.

(a) $h_1[n] = h[1-n]$,

(b) $h_2[n] = \left(2e^{j\pi/4}\right)^n h[n]$,



4.- A causal LTI system has system function given by the following expression:

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1 + 2z^{-1}}{1 + \frac{3}{4}z^{-1} - \frac{1}{4}z^{-2}}.$$

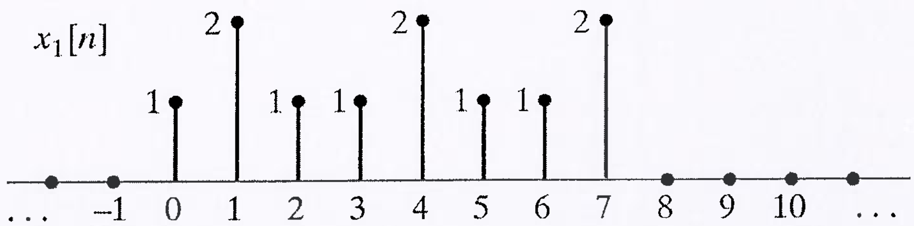
(a) Is this system stable? Explain briefly. [3 pts]

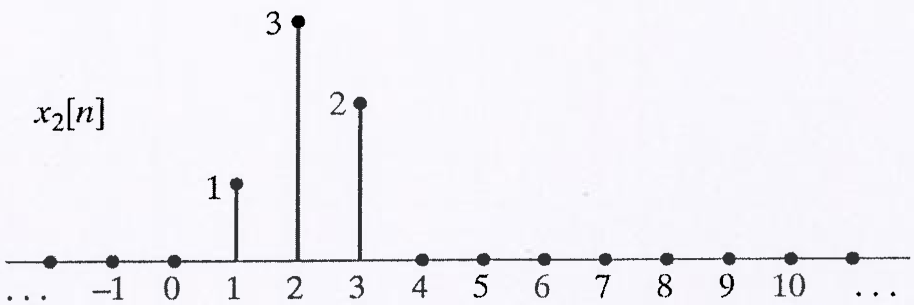
(b) Draw the signal flow graph of a parallel form implementation of this system. Use a direct form II implementation for the 2^{nd}-order subsystem. [3 pts]

(c) Draw the signal flow graph of a cascade form implementation of this system as a cascade of a 1^{st}-order system and a 2^{nd}-order system. Use a transposed direct form II implementation for the 2^{nd}-order system. [3 pts]

(d) Considering the number of storage elements and multiplications required to produce the system output using the implementation structures proposed in parts (b) and (c), which one of the two is preferable? Justify your answer. [3 pts]

5.- Two finite-length signals, $x_1[n]$ and $x_2[n]$, are sketched in the figure below. Assume that $x_1[n]$ and $x_2[n]$ are zero outside of the interval shown in the figure. Let $x_3[n]$ be the eight-point circular convolution of $x_1[n]$ with $x_2[n]$; i.e., $x_3[n] = x_1[n]\ 8\ x_2[n]$.





(a) Determine $x_3[n]$. [4 pts]

Let $x_4[n]$ be the linear convolution of $x_1[n]$ with $x_2[n]$.

(b) What is the length of $x_4[n]$? [2 pts]

(c) Explain how $x_3[n]$ and $x_4[n]$ are related. Provide this relationship. [3 pts]

(d) You are given a radix-2 FFT algorithm (bit-reversal is built-in).
Describe how you would use it to compute $x_4[n]$ from $x_1[n]$ and $x_2[n]$. [3 pts]
i) How many times this FFT algorithm would need to be used?
ii) Specify the number of points of each FFT required.
iii) How are inputs $x_1[n]$ and $x_2[n]$ fed to the FFT algorithm(s)?
Is zero padding needed?
Suggestion: Sketch a diagram showing the steps required to obtain $x_4[n]$.

6.- (a) During the design of an IIR digital filter using the bilinear transformation, the s-plane poles obtained in the design of the Butterworth analogue filter prototype based on $|H(j\Omega)|^2$ are:

$p_1 = 0.71e^{j\pi/12}$ $p_5 = 0.71e^{j9\pi/12}$ $p_9 = 0.71e^{j17\pi/12}$

$p_2 = 0.71e^{j3\pi/12}$ $p_6 = 0.71e^{j11\pi/12}$ $p_{10} = 0.71e^{j19\pi/12}$

$p_3 = 0.71e^{j5\pi/12}$ $p_7 = 0.71e^{j13\pi/12}$ $p_{11} = 0.71e^{j21\pi/12}$

$p_4 = 0.71e^{j7\pi/12}$ $p_8 = 0.71e^{j15\pi/12}$ $p_{12} = 0.71e^{j23\pi/12}$

(i) Which poles would you choose to form the analogue filter response $H(s)$? Explain why. [2 pts]

(ii) How would you then find the digital filter response $H(z)$ from $H(s)$? [2 pts]

(b) You are to design a lowpass IIR digital filter using the impulse invariance transformation. During the design of the Butterworth analogue filter prototype you solve a system of two equations determined by the desired frequency response for the pass-band corner frequency and the stop-band corner frequency.

The values resulting for the Butterworth analogue filter parameters are:

$\Omega_c = 0.815$, and $N = 5.305$

(i) What should your next step be? [2 pts]

(ii) If you were to exactly meet the desired frequency response specifications for the passband corner frequency or the stopband corner frequency, which one of the two would you choose to meet? Explain why. [2 pts]

(c) There are four types of linear-phase FIR digital filters. Not all of them are able to accommodate all types of frequency selective filters; lowpass, highpass, bandpass and bandstop.

In the table below mark with an X those implementations that are not possible to occur and justify why not based on the z-plane zero locations of $H(z)$ for those filter types. [4 pts]

FIR Filter	Lowpass	Highpass	Bandpass	Bandstop
Type I				
Type II				
Type III				
Type IV				

Additional Information

(Not all of this information is necessarily required today!)

DTFT Synthesis Equation	DTFT Analysis Equation
$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$	$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$
Parseval's Theorem $E = \sum_{n=-\infty}^{\infty}\lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\lvert X(e^{j\omega})\rvert^2 d\omega$	**DFT Exponential Factor** $W_N = e^{-j(2\pi/N)}$
Z-transform of a sequence $x[n]$ $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$	**Z-transform Properties** $x[-n] \overset{Z}{\leftrightarrow} X(1/z),\ ROC = \frac{1}{R_x}$ $z_o^n x[n] \overset{Z}{\leftrightarrow} X(z/z_o),\ ROC = \lvert z_o\rvert R_x$

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n - n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0\rvert R_x$
4	3.4.4	$nx[n]$	$-z\frac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{Re}\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
7		$\mathcal{Im}\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Properties of the Discrete Fourier Transform

	Finite-Length Sequence (Length N)	N-point DFT (Length N)
1.	$x[n]$	$X[k]$
2.	$x_1[n], x_2[n]$	$X_1[k], X_2[k]$
3.	$ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
4.	$X[n]$	$Nx[((-k))_N]$
5.	$x[((n-m))_N]$	$W_N^{km} X[k]$
6.	$W_N^{-\ell n} x[n]$	$X[((k-\ell))_N]$
7.	$\sum_{m=0}^{N-1} x_1(m) x_2[((n-m))_N]$	$X_1[k]X_2[k]$
8.	$x_1[n]x_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1} X_1(\ell) X_2[((k-\ell))_N]$
9.	$x^*[n]$	$X^*[((-k))_N]$
10.	$x^*[((-n))_N]$	$X^*[k]$
11.	$\mathcal{R}e\{x[n]\}$	$X_{ep}[k] = \frac{1}{2}\{X[((k))_N] + X^*[((-k))_N]\}$
12.	$j\mathcal{J}m\{x[n]\}$	$X_{op}[k] = \frac{1}{2}\{X[((k))_N] - X^*[((-k))_N]\}$
13.	$x_{ep}[n] = \frac{1}{2}\{x[n] + x^*[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$
14.	$x_{op}[n] = \frac{1}{2}\{x[n] - x^*[((-n))_N]\}$	$j\mathcal{J}m\{X[k]\}$
	Properties 15–17 apply only when $x[n]$ is real.	
15.	Symmetry properties	$\begin{cases} X[k] = X^*[((-k))_N] \\ \mathcal{R}e\{X[k]\} = \mathcal{R}e\{X[((-k))_N]\} \\ \mathcal{J}m\{X[k]\} = -\mathcal{J}m\{X[((-k))_N]\} \\ \lvert X[k]\rvert = \lvert X[((-k))_N]\rvert \\ \sphericalangle\{X[k]\} = -\sphericalangle\{X[((-k))_N]\} \end{cases}$
16.	$x_{ep}[n] = \frac{1}{2}\{x[n] + x[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$
17.	$x_{op}[n] = \frac{1}{2}\{x[n] - x[((-n))_N]\}$	$j\mathcal{J}m\{X[k]\}$

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $[\cos\omega_0 n]u[n]$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

<u>Initial</u> <u>Value</u> <u>Theorem</u>:

If $x[n]$ is a causal sequence, *i.e.* $x[n] = 0$, $\forall n < 0$, then

$$x[0] = \lim_{z\to\infty} X(z)$$