National Examination - 2016

04-BS-16, Discrete Mathematics

Duration: 3 hours

Examination Type: Closed Book, One of two calculators is permitted - any Casio of Sharp approved model

Last Name:

First Name:

Do not turn this page until the permission is given. Meanwhile read the instructions below.

If doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper a clear statement of any assumption made.

Instructions:

- This exam paper contains 13 pages (including this cover page).
- **You have to answer 10 questions out of 12.**
- Clearly indicate which questions you do not want to answer both on the cover page by crossing it and on the corresponding page by drawing a diagonal line across the page.

Question:	1	2	3	4	5	6	7	8	9	10	11	12	Total
Points:	10	10	10	10	10	10	10	10	10	10	10	10	100
Score:													

1. Answer the following questions on propositions and their relations.

(a) $\boxed{\text{3 points}}$ Prove that the following proposition is a tautology: $\neg(\neg p \vee q) \rightarrow p$.
(Note: $\neg p$ is the negation of p.)

(b) $\boxed{\text{2 points}}$ Determine the truth value of the following proposition "If $x = x + 5$ then $2x = 2x - 3$."

(c) $\boxed{\text{2 points}}$ Write the negation of the proposition "$\forall x \quad x^2 \geq 2x - 1$".

(d) $\boxed{\text{3 points}}$ Write an equivalent proposition for $(p \vee \neg q) \wedge (\neg p \vee \neg q)$ in the simplest form that you can.

2. Answer the following questions related to truth of propositions.

(a) $\boxed{\text{3 points}}$ Write the truth table for the compound proposition $\neg p \to (p \wedge \neg q)$.

(b) $\boxed{\text{3 points}}$ Determine the truth value of "$\forall x \quad x^2 \geq 2x - 1$" where the universe of discourse is the set of real numbers.

(c) $\boxed{\text{4 points}}$ Determine whether $\forall x(P(x) \to Q(x))$ has the same truth value as $\forall x\, P(x) \to \forall x\, Q(x)$.

3. Answer the following questions related to set theory.

(a) $\boxed{\text{3 points}}$ If $A = \{0, 1, 2, 3\}$ and $B = \{1, 3, 4, 5\}$, find $(B - A) \times (A - B)$.

(b) $\boxed{\text{3 points}}$ Let A, B and C be sets. Using algebra of sets show that $(A \cap B) - C^c = (B \cap C) - A^c$.

(c) $\boxed{\text{4 points}}$ Let A, B and C be three sets such that $A \cap C = B \cap C$ and $A \cup C = B \cup C$. Prove that $A \cap C^c = B \cap C^c$.

4. Answer the following questions related to functions.

(a) $\boxed{\text{2 points}}$ Determine if $f(n) = \sqrt{n^3+1}$ is a function from $\mathbb{Z}$ to $\mathbb{R}$.

(b) $\boxed{\text{2 points}}$ Find the range of function $f(x) = \sqrt{x^4+1}$ if the domain of f is $x \in \mathbb{R}$.

(c) $\boxed{\text{2 points}}$ Determine if the function $f:\ \mathbb{R} \to \mathbb{R}$ and $f(x) = 2x^4+1$ is one-to-one.

(d) $\boxed{\text{4 points}}$ Consider functions f and g both defined from $\mathbb{R}$ to $\mathbb{R}$. Also $f(x) = x^3+2$ and $(f+g)(x) = x^3+x+2$. First find $g(x)$ then use it to find $(g \circ f^{-1})^{-1}(x)$.

5. Answer the following questions on relations.

Consider the relation R defined on set $\{-1,0,1,2\}$ where $(x,y) \in R$ if and only if $x = y \pm 1$.

(a) $\boxed{\text{2 points}}$ Write all elements of R.

(b) $\boxed{\text{3 points}}$ Determine if R is reflexive, symmetric, antisymmetric and/or transitive.

(c) $\boxed{\text{3 points}}$ Write all elements of R^2?

(d) $\boxed{\text{2 points}}$ Determine if R^2 is reflexive.

6. This is a question on counting.

   Consider the permutations of the letters of the word **TORONTONIANS**.

   (a) $\boxed{\text{2 points}}$ How many are there in total?

   (b) $\boxed{\text{2 points}}$ How many start with T and end with S?

   (c) $\boxed{\text{2 points}}$ How many start with a vowel?

   (d) $\boxed{\text{2 points}}$ How many have the block **NOTNOT**, with these six letters appearing in this order?

   (e) $\boxed{\text{2 points}}$ How many have the same letter at the first and last position.?

7. Answer the following questions related to discrete probability.

   (a) $\boxed{\text{5 points}}$ A fair six-sided die is rolled twice. Find the probability that the first roll be greater than or equal to the second roll.

   (b) $\boxed{\text{5 points}}$ In a city, 45% of men and 60% of women have blond hair. The rest of the population has dark hair. Also, 52% of the population are female and 48% male. A random person is known to have dark hair. Find the probability that this person is a male.

8. This is a question on series and sequences.

Let S_n be the number of all n-bit binary strings that do not contain the pattern 11. Also, let V_n be the number of n-bit binary strings that start with 0 and do not contain 11. Finally, let W_n be the number of n-bit binary strings that start with 10 and do not contain 11.

(a) $\boxed{\text{1 point}}$ Find S_1 and S_2 by checking all possible sequences.

(b) $\boxed{\text{2 points}}$ Write V_n and W_n in terms of S_{n-1} and S_{n-2}.

(c) $\boxed{\text{3 points}}$ Use part (b) to write S_n in terms of S_{n-1} and S_{n-2}.

(d) $\boxed{\text{4 points}}$ Use parts (a) and (c) to find a closed-form formula for S_n in terms of n.

9. This is a question on methods of proof.

(a) $\boxed{\text{5 points}}$ Without using induction, prove that for any positive integer n , $n^3 + 2n$ is divisible by 3.

(b) $\boxed{\text{5 points}}$ Use induction to prove that for any positive integer n, $3^n > n^2$.

10. This is another question on methods of proof.

(a) $\boxed{\text{5 points}}$ Show that at least eight of any 50 days must fall on the same day of the week.

(b) $\boxed{\text{5 points}}$ Show that for real numbers x and y, $|x-1|+|y+1| \geq |x+y|$.

11. This is a question on growth of functions and complexity of algorithms

(a) $\boxed{\text{4 points}}$ Show that $f(n) = 10\log(n!)$ is $O(n\log n)$.

(b) $\boxed{\text{3 points}}$ The time complexity of Algorithms A and B are $\Theta(n^2)$ and $\Theta(n\log n)$ respectively. Can it be said that on a problem with size $n = 10$, it is certain that Algorithm A takes a longer time than B? Justify your answer.

(c) $\boxed{\text{3 points}}$ In part (b), can it be said that there exist some n^* such that for any problem with size larger than n^*, Algorithm A takes longer than B? Justify your answer.

12. This is a question on graphs theory.

(a) $\boxed{\text{2 points}}$ Is it possible to construct a graph such that it has 25 vertices all with degree 5? Justify your answer.

(b) $\boxed{\text{2 points}}$ Let K_n be the complete graph with n vertices for $n \geq 1$. For what values of n, K_n is planar?

(c) $\boxed{\text{2 points}}$ The adjacency matrix of graph G (with vertices a,b,c in the same order) is A such that

$$A^3 = \begin{bmatrix} 4 & 5 & 2 \\ 5 & 6 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

How many paths of length 3 exists between vertices b and c?

(d) $\boxed{\text{4 points}}$ Let $K_{m,n}$ denote the complete bipartite graph with m vertices on one side and n vertices on the other side. For what values of m and n does $K_{m,n}$ have an Euler path?