National Examination - May 2019

04-BS-16, Discrete Mathematics

Duration: 3 hours

NOTES:

1. If doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper a clear statement of any assumptions made.

2. This is a CLOSED BOOK exam.

3. An approved Casio or Sharp calculator is acceptable.

4. Answer 10 of the 12 questions.

5. Clearly indicate which questions you do not want to answer both below on this page and on the corresponding exam page by drawing a diagonal line through it.

Question:	1	2	3	4	5	6	7	8	9	10	11	12	Total
Points:	10	10	10	10	10	10	10	10	10	10	10	10	100
Score:													

1. Answer the following questions on propositions and their relations.

   (a) 3 points Write the negation of the proposition "$\exists x \quad 2x^2 \leq x - 1$" using the identifier $\forall$.

   (b) 3 points Write the truth table for the proposition $(\neg r \vee q) \rightarrow (p \vee r)$.
   (Note: $\neg s$ is the negation of s.)

   (c) 4 points Determine whether $\forall x(P(x) \rightarrow Q(x))$ has the same truth value as $\forall x\, P(x) \rightarrow \forall x\, Q(x)$.

2. Answer the following questions related to truth of propositions.

   (a) [3 points] Determine the truth value of "$\forall n \quad 4n^2 > 4n - 1$" where the universe of discourse is the set of integers.

   (b) [2 points] Determine the truth value of the following proposition "If $2 > 5$ then $\forall x, x^2 < x^2 - 5$."

   (c) [2 points] Write an equivalent proposition for $(\neg p \vee q) \wedge (p \vee q)$ in the simplest form that you can.

   (d) [3 points] Prove the following proposition is an tautology: $(\neg(p \vee \neg q) \wedge \neg q) \rightarrow r$.

3. Answer the following questions related to set theory.

(a) [3 points] If $A = \{0,1,2,3\}$ and $B = \{1,3,4,5\}$, find $(A \cap B) \times (A - B)$.

(b) [4 points] Let A, B and C be sets. Using algebra of sets show that $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

(c) [3 points] Let A, B be two sets, prove that $B \subset (A - B)^c$.

4. Answer the following questions related to discrete probability.

(a) [5 points] A fair six-sided die is rolled twice. Find the probability that the sum of the two rolls is 9.

(b) [5 points] A TV factory has three production lines A,B and C. Lines A,B and C produce 35%, 50% and 15% of the total respectively. Also, of the outputs of lines A, B and C, 2%, 5% and 1% are defective, respectively. A random TV from this company is found the be defective. Find the probability that it was produced by line C.

5. Answer the following questions related to functions.

(a) [2 points] Determine if $f(n) = \sqrt{n^2 - 9}$ is a function from $\mathbb{Z}$ to $\mathbb{R}$.

(b) [2 points] Find the range of function $f(x) = \sqrt{x^2 + 16}$ if the domain of f is $x \in \mathbb{R}$.

(c) [2 points] Determine if the function $f: \mathbb{R} \to \mathbb{R}$ and $f(x) = 5x^5 + 5$ is one-to-one.

(d) [4 points] Consider functions f and g both defined from $\mathbb{R}$ to $\mathbb{R}$. Also $f(x) = x^5$ and $(f+g)(x) = x^5 + x^3$. Find $f^{-1}(x)$ and $g \circ f(x)$.

6. This is a question on counting.

   Consider the permutations of the letters of the word **ENGINEERING**.

   (a) [2 points] How many are there in total?

   (b) [2 points] How many start with R and end with G?

   (c) [2 points] How many start with a vowel?

   (d) [2 points] How many have all the E's together as "EEE"?

   (e) [2 points] How many start with ING and end with ING.?

7. This is a question on series and sequences.

(a) 5 points Prove that the sum of all elements of the set $A = \{1, 5, 9, ..., 4n-1\}$ is $S = 2n^2 - n$.

(b) 5 points Let us define $a_1 = 2$ and $a_i = a_{i-1} + 2i - 1$. Prove that $a_n = n^2 + 1$.

8. This is a question on methods of proof.

(a) [5 points] Without using induction, prove that for any positive integer n , $n^3 - n$ is divisible by 3.

(b) [5 points] Use induction to prove that for any integer $n > 2$, $4^n > n^4$.

9. This is another question on methods of proof.

(a) 5 points Show that among 1100 people, at least four must have the same birthday.

(b) 5 points Prove that for real numbers x and y, $|x-3|+|y+3| \geq |x+y|$.

10. Answer the following questions on relations.

Consider the relation R defined on set $A = \{-2, -1, 0, 1, 2\}$ where $(x, y) \in R$ if and only if $x = -y \pm 1$.

(a) 2 points Write all elements of R.

(b) 3 points Determine if R is reflexive, symmetric, antisymmetric and/or transitive.

(c) 3 points Write all elements of R^2?

(d) 2 points Give a relation on A that is symmetric and antisymmetric.

11. This is a question on growth of functions and complexity of algorithms
The time complexity of Algorithms A and B are $\Theta(n^3)$ and $\Theta(4^n)$ respectively.

(a) 3 points Can it be said that on a problem with size $n = 4$ Algorithm B takes a longer time than A? Justify your answer.

(b) 3 points Now, consider two problems with sizes k and $2k$ respectively, where k is large. Can it be said that Algorithm A takes approximately eight times longer on the larger problem? Justify your answer.

(c) 4 points Show that $f(n) = 5n + 3\log(n!)$ is $O(n\log n)$.

12. This is a question on graphs theory.

(a) 3 points Is it possible to construct a graph such that it has 13 vertices all with degree 7? Justify your answer.

(b) 2 points The adjacency matrix of graph G (with vertices a,b,c in the same order) is A such that

$$A^5 = \begin{bmatrix} 4 & 2 & 4 \\ 2 & 12 & 7 \\ 4 & 7 & 6 \end{bmatrix}$$

How many paths of length 5 exists between vertices b and c?

(c) 2 points A connected bipartite graph has 7 verticies on one side (left side) and 5 degree two vertices on the other side (right side). What is the maximum possible degree on the left side?

(d) 3 points Let K_n be the complete graph with n vertices for $n \geq 1$. For what values of n, K_n is planar?