Please put all your answers directly on the exam paper with the space provided. Do not use an exam booklet. If more space is required please use the back of the exam page and clearly number the exam question response to *reflect the appropriate exam question* answered.

National Examination – May 2013
04-BS-16: Discrete Mathematics
Duration: 3 hours

Examination Type: Closed Book.
No aids allowed.

Last Name: ______________________________

First Name: ______________________________

Do not turn this page until you have received the signal to start.
(In the meantime, please fill out the identification section above, and read the instructions below.)

# 1:	______/ 10
# 2:	______/ 10
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# 9:	______/ 10
# 10:	______/ 10
# 11:	______/ 10
# 12:	______/ 10
TOTAL:	______/100

This exam paper contains 13 pages (including this one).
Answer 10 out of 12 questions. Ten questions constitute a full paper.
Please clearly indicate which two questions you don't want marked by drawing a diagonal line across the page.
In case of doubt to any question, clearly state any assumptions made.

Good Luck!

Question 1. [10 MARKS]

Logic: Propositions, compound propositions and quantifiers.

a. Restate each proposition in the form "If ... then ...":
   X is sufficient for Y.

   Y only if X.

b. Suppose proposition q is true and proposition p has an unknown status. Is the compound proposition $((q \wedge p) \rightarrow p) \wedge (p \rightarrow (q \wedge p))$ true or false? Prove using a truth table.

c. Let $P(x, y)$ be the propositional function $x \leq y$. The domain of discourse is the set of all real numbers. For each proposition, state whether it is true or false. Then, use the generalized De Morgan's Laws to simplify the negation of each proposition to a form where the negation symbol is not used.

   $\forall x \, \forall y \, P(x, y)$

   $\exists x \, \exists y \, P(x, y)$

Question 2. [10 MARKS]

Part (a) [5 MARKS]

Use mathematical *induction* to prove that for any integer x, where $x > 1$, $x^n - 1$ is divisible by $x - 1$, for $n = 1, 2, 3, \ldots$.

Part (b) [5 MARKS]

Consider a sequence defined by: $a_1 = 1$, $a_2 = 2$, and

$$a_n = 2a_{n-1} + 3a_{n-2}, \quad \text{if } n \geq 3$$

Prove that $\forall n > 1$, $a_n \leq 5^{n-1}$.

Question 3. [10 MARKS]

Part (a) [5 MARKS]

Prove that any amount of postage of 12 cents or more can be formed using only 4-cent and 5-cent stamps.

Part (b) [5 MARKS]

Prove that if integers $1, 2, \cdots, 10$ are put into three groups, there exists at least one group whose sum is greater than or equal to 19.

Question 4. [10 MARKS]

Part (a) [4 MARKS]

Give the power set of $\{a, b, c\}$. List all partitions of the set $\{a, b, c\}$.

Part (b) [6 MARKS]

Let $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ be positive. Let $S(\alpha, \beta)$ be families of sets in $\mathbb{R}^2$ defined as follows:

$$S(\alpha,\beta) \quad = \quad \{(x,y) | -\alpha \geq x \geq \alpha, -\beta \geq y \geq \beta\}$$

a. Briefly describe the set $S(\alpha, \beta)$ in plain English.

b. What is $\displaystyle\bigcup_{\alpha>0,\beta>0} S(\alpha,\beta)$?

c. What is $\displaystyle\bigcap_{\alpha>0,\beta>0} S(\alpha,\beta)$?

Question 5. [10 MARKS]

Consider the relation R on $\{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x^2 \geq y$.

a. Is R reflexive? Justify your answer.

b. Is R symmetric? Justify your answer.

c. Is R anti-symmetric? Justify your answer.

d. Is R transitive? Justify your answer.

e. Is R an equivalence relation? Justify your answer.

Question 6. [10 MARKS]

Part (a) [4 MARKS]

Find an expression in terms of n, $n \geq 1$, for *exactly* how many times the floor operation $\lfloor \cdot \rfloor$ is executed in the following algorithm.

```
i=n
while(i>=1){
  i = ⌊i/2⌋
}
```

Part (b) [3 MARKS]

What does it mean that $f(n) = O(g(n))$? Show that $n! = O(n^n)$.

Part (c) [3 MARKS]

What does it mean that $f(n) = \Omega(g(n))$? Show that $n! = \Omega(2^n)$.

Question 7. [10 MARKS]

You may express answers in factorial form in this question.

Part (a) [7 MARKS]

In how many ways can all the letters of the word STATEMENTS be arranged if

a. there are no restrictions?

b. all the T's must be together?

c. none of the T's can be together?

Part (b) [3 MARKS]

Given the equation $x_1 + x_2 + x_3 + x_4 = 11$, find the number of integer solutions if we require $x_i \geq 0$ for $i = 1, 2, 3, 4$.

Question 8. [10 MARKS]

Part (a) [6 MARKS]

An experiment consists of throwing three fair dice. Find the probability that

a. all three dice show the same value

b. all three dice show different values

c. at least one die shows a six

Part (b) [4 MARKS]

Three pairs of twins go to the theatre and sit in three separate pairs of seats. If the tickets are mixed up and each person is assigned to one of the six seats at random. Find the probability that everyone sits beside his own twin.

Question 9. [10 MARKS]

Part (a) [6 MARKS]

Consider K_6, the *complete graph on six* vertices. Suppose you colour each edge either red or blue. Use the pigeonhole principle to prove that the resulting graph contains at least one triangle with three red sides or three blue sides.

Part (b) [4 MARKS]

A high school offers three math courses: Calculus, Algebra and Geometry. Among 120 students, it is found that 15 did not take any math courses, while 25 took all three. Further, 35 took Calclus and Algebra; 45 took Calculus and Geometry; and 25 took Algebra and Geometry. How many students took exactly one math course?

Question 10. [10 MARKS]

Part (a) [2 MARKS]

What is an Euler path? What is a Hamilton path?

Part (b) [2 MARKS]

Under what conditions does a simple connected graph have an Euler path from vertex u to vertex v?

Part (c) [2 MARKS]

Determine whether or not $K_{2,3}$ is planar. ($K_{2,3}$ is the complete bipartite graph with 2 vertices on one side and 3 vertices on the other.) Justify.

Part (d) [2 MARKS]

What problem does Dijkstra's algorithm solve?

Part (e) [2 MARKS]

What is the chromatic number of K_5 (the complete graph on six vertices)?

Question 11. [10 MARKS]

Part (a) [3 MARKS]

A finite, connected, planar graph is drawn in the plane without any edge intersections. Let v be the number of vertices, e be the number of edges, and f be the number of faces (i.e., regions bounded by edges, including the outer, infinitely large region). Write down a mathematical relationship between v, e and f.

Part (b) [7 MARKS]

A polyhedron has twelve pentagons and twenty hexagons for its faces. At every vertex exactly two hexagons and one pentagon meet. How many vertices and how many edges does this polyhedron have?

Question 12. [10 MARKS]

Part (a) [4 MARKS]

Assuming that, in all three-child family, all eight possible sequences of male/female children are equally likely, find the probability that a family has two boys and one girl given that there is at least one boy.

Part (b) [6 MARKS]

Suppose that we generates a byte (8 bits) at random. Each bit has 50% chance of being 1 or 0. Further, each bit is generated independently of other bits. (i) What is the probability that the byte has exactly three 1's? (ii) What is the probability that the byte has at least three 1's? (The 1's don't necessarily need to be consecutive.)

Total Marks = 100