Professional Engineers Ontario

National Examinations December 2019

16-Civ-B9, The Finite Element Method

3 hours duration

Notes:

1. There are 4 pages in this examination.
2. If doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper, a clear statement of any assumptions made
3. This is a closed book exam but one aid sheet allowed written on both sides.
4. Candidates may use a Casio or Sharp approved calculator.
5. **Answer all proposed problems**
6. Marking scheme is shown on the exam paper.

Problem 1 (25 Marks)

A three-bar truss is shown in Figure 1. All bars have the same axial stiffness EA. When the temperature was at a reference value T_0, the truss was subjected to a vertical force P. The temperature of bar **1-2** is increased by an amount ΔT above T_0. The thermal coefficients of all bars are equal and denoted α.

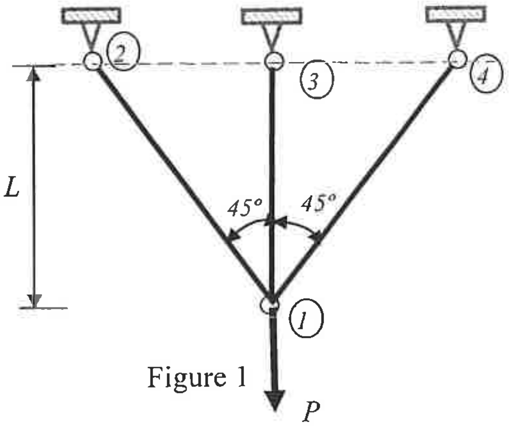


Figure 1

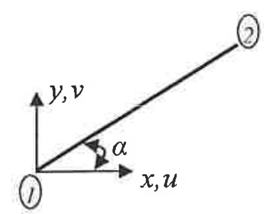
$L = 2000\ mm,\ A = 3000\ mm^2$
$E = 200\ GPa$
$\Delta T = 30^o,\ \alpha = 12 \times 10^{-6}/°C$

1.1 Determine the displacement of joint 1 and the forces in the three bars when the system reaches equilibrium.

1.2 Using this example as a hint, explain how you proceed to analyse a 3D structure subjected to a mechanical force system as well as a temperature gradient.

The stiffness matrix of a bar with orientation defined by an angle α with respect to the global reference system (x, y) is given by:

$$[k] = \frac{EA}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix} \qquad C = \cos(\alpha),\ \ S = \sin(\alpha)$$



The internal force in the bar is given by:

$$F_{(1,2)} = \frac{EA}{L}\begin{bmatrix} C & S \end{bmatrix}\begin{bmatrix} u_2 - u_1 \\ v_2 - v_1 \end{bmatrix}$$

where u_1, v_1 are the displacement of node 1 in the global reference

and u_2, v_2 are the displacement of node 2 in the global reference

Problem 2 (25 marks)

A beam is connected to two truss bars is shown in Figure 2. The beam is considered inextensible (axial rigidity is infinite) with bending rigidity *EI*. The truss bars are assumed to have hinge connections at the both ends. The axial rigidity *EA* of both truss bars is assumed to be equal to $24\frac{EI}{L^2}$.

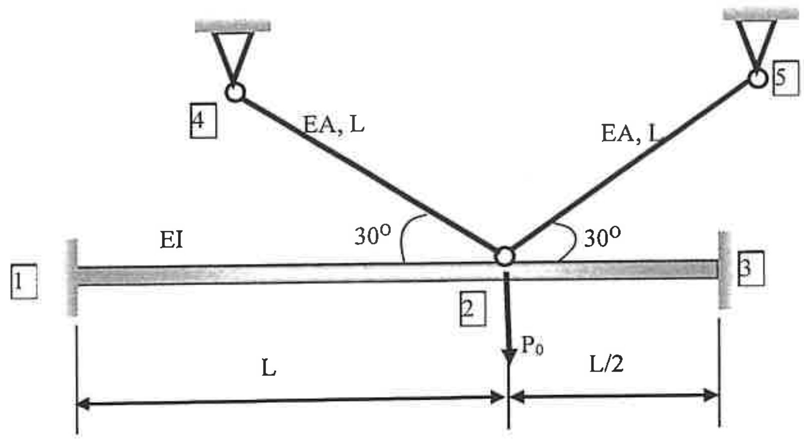


Figure 2

2.1 Using a discretization with two beam and two truss elements, define the active DOF of the system.
2.2 Assemble the stiffness matrix of the system
2.3 Calculate the displacements in the system.
2.4 Calculate the axial forces in the two truss bars.

The stiffness matrix of a beam and truss elements are given by:

$$[K] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ SYM & & & 4l^2 \end{bmatrix}$$

Problem 3 (50 marks)

3.1 Referring to Figure 3(a), the interpolation of the displacement field inside the Q4 element is given by:

$$u(x,y) = \sum_{i=1}^{4} N_i(x,y)u_i \quad \text{and } v(x,y) = \sum_{i=1}^{4} N_i(x,y)v_i$$

Show that the shape function N_i are given by:

$$N_1 = \left(1-\frac{x}{L}\right)\left(1-\frac{y}{L}\right);\ N_2 = \frac{x}{L}\left(1-\frac{y}{L}\right);\ N_3 = \frac{xy}{L^2};\ N_4 = \left(1-\frac{x}{L}\right)\frac{y}{L}$$

3.2 Derive the strain-displacement matrix [B] defined from:

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = [B]\{d\} \text{ and vector } \{d\} \quad = \{u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4\}^T$$

3.3 Show that in the case of plane stress, the linear relationship between the horizontal forces $\{F_2, F_3\}$ and displacements of node 2 and 3, $\{u_2, u_3\}$ is given by:

$$\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

where $k_{11} = k_{22} = \frac{Et}{1-v^2}\left[\frac{1}{3} + \frac{1-v}{6}\right], k_{12} = \frac{Et}{1-v^2}\left[\frac{1}{6} + \frac{1-v}{6}\right]$. t is the thickness of the element. The element is made of an elastic material with an elasticity modulus E and Poisson's ratio v.

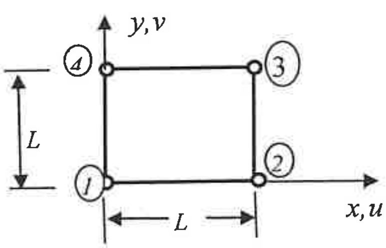


Fig. 3(a)

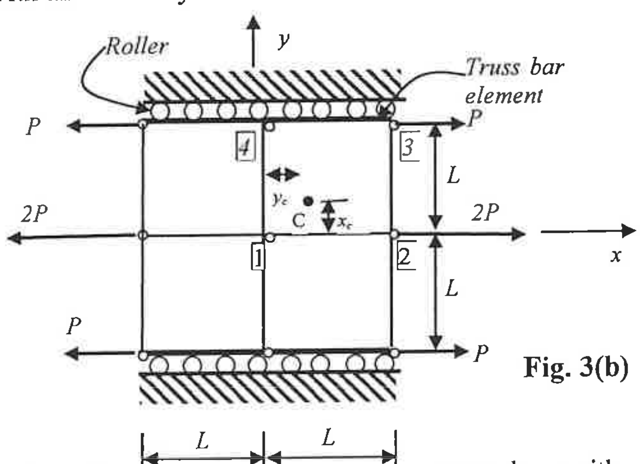


Fig. 3(b)

3.4 Figure 3(b) shows a plate of thickness t reinforced at two bars with constant cross sections A. The plate is modeled using four rectangular elements. Assuming that $Et = \frac{2EA}{L}$.

Calculate displacements at node 2 and 3.

Calculate the strains at point C, $x_c = y_c = \frac{L}{4}$.

Use the following data: $E = 200\ GPa$, $\nu = 0.25$, $L = 20\ mm$, $t = 5\ mm$ and $P = 5\ kN$.