NATIONAL EXAMINATIONS DECEMBER 2018

04-BS-5 ADVANCED MATHEMATICS

3 Hours duration

NOTES:

1. If doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper, a clear statement of any assumption made.

2. Candidates may use one of the approved Casio or Sharp calculators. This is a Closed Book Exam. However, candidates are permitted to bring **ONE** aid sheet (8.5"X11") written on both sides.

3. Any five (5) questions constitute a complete paper. Only the first five answers as they appear in your answer book will be marked.

4. All questions are of equal value.

Marking Scheme

1. 20 marks
2. (a) 15 marks ; (b) 5 marks
3. (a) 4 marks ; (b) 10 marks ; (c) 6 marks
4. (a) 10 marks ; (b) 10 marks
5. (a) 17 marks ; (b) 3 marks
6. (A) (a) 6 marks ; (b) 7 marks; (B) 7 marks
7. (a) 10 marks ; (b) 10 marks

1. Find the eigenvalues and eigenfunctions of the following regular Sturm-Liouville problem:

$$\frac{d}{dx}\left(x^{-3}\frac{dy}{dx}\right) + (4+\lambda)x^{-5}y = 0\,; \quad y(1) = y(e^2)$$

2. (a) Find the Fourier series expansion of the periodic function F(x) of period $p=4\pi$.

$$F(x) = x^2 \quad ; \qquad -2\pi \le x \le 2\pi$$

(b) Use the result obtained in (a) to find the Fourier series expansion of the periodic function G(x) of period $p=4\pi$.

$$G(x) = x \quad ; \qquad -2\pi < x < 2\pi$$

3. Consider the following function where τ is a positive constant

$$f(x) = \begin{cases} (1 + x/4\tau)/\tau & -4\tau \le x < 0 \\ (1 - x/4\tau)/\tau & 0 \le x \le 4\tau \end{cases}$$

Note that $f(x) = 0$ for all the other values of x.

(a) Compute the area bounded by f(x) and the x-axis. Graph f(x) against x for $\tau = 0.5$ and $\tau = 0.25$ on the same set of axes.

(b) Find the Fourier transform $F(\omega)$ of f(x).

(c) Graph $F(\omega)$ against ω for the same two values of τ mentioned in (a). Explain what happens to f(x) and $F(\omega)$ when τ tends to zero.

Note: $$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\exp(-i\omega x)\,dx$$

4.(A) Prove that the coefficients α and β of the least-squares parabola $Y = \alpha X + \beta X^2$ that fits the set of n points (X_i, Y_i) can be obtained as follows

$$\alpha = \frac{\left\{\sum_{i=1}^{i=n} X_i Y_i\right\}\left\{\sum_{i=1}^{i=n} X_i^4\right\} - \left\{\sum_{i=1}^{i=n} X_i^2 Y_i\right\}\left\{\sum_{i=1}^{i=n} X_i^3\right\}}{\left\{\sum_{i=1}^{i=n} X_i^2\right\}\left\{\sum_{i=1}^{i=n} X_i^4\right\} - \left\{\sum_{i=1}^{i=n} X_i^3\right\}^2};$$

$$\beta = \frac{\left\{\sum_{i=1}^{i=n} X_i^2\right\}\left\{\sum_{i=1}^{i=n} X_i^2 Y_i\right\} - \left\{\sum_{i=1}^{i=n} X_i^3\right\}\left\{\sum_{i=1}^{i=n} X_i Y_i\right\}}{\left\{\sum_{i=1}^{i=n} X_i^2\right\}\left\{\sum_{i=1}^{i=n} X_i^4\right\} - \left\{\sum_{i=1}^{i=n} X_i^3\right\}^2}$$

4.(B) Use the method of Lagrange to find the third degree polynomial that fits the following set of four points.

x	–3	–2	0	2
F(x)	0	4	–6	0

5.(A) The following results were obtained in a certain experiment.

x	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
F(x)	10.00	63.75	70.00	86.25	80.00	68.75	60.00	61.25	90.00
G(x)	6.00	47.50	53.00	62.50	66.00	58.50	41.00	41.50	64.00

Use Romberg's algorithm to evaluate the area bounded by the unknown function G(x) given in the table and the lines x=0, x=4.0 and the x-axis.

Hint: The Romberg algorithm produces a triangular array of numbers, all of which are numerical estimates of the definite integral $\int_a^b f(x)dx$. The array is denoted by the following notation.

R(1,1)			
R(2,1)	R(2,2)		
R(3,1)	R(3,2)	R(3,3)	
R(4,1)	R(4,2)	R(4,3)	R(4,4)

where

$$R(1,1) = \frac{H_1}{2}[f(a) + f(b)]$$

$$R(k,,1) = \frac{1}{2}\left[R(k-1,1) + H_{k-1}\sum_{n=1}^{n=2^{k-2}} f(a + (2n-1)H_k)\right]; \qquad H_k = \frac{b-a}{2^{k-1}}$$

$$R(k,j) = R(k,j-1) + \frac{R(k,j-1) - R(k-1,j-1)}{4^{j-1} - 1}$$

5.(B) Application of Romberg's algorithm in finding the area bounded by the function F(x). and the lines x=0, x=4.0 and y = 0 yielded the result R(4,4) = 274.476180. Use this result and the result obtained in (A) to find the area bounded by the functions F(x), G(x), x=0 and x=4.0.

6.(A)(a) One root of the equation $3^x + x^2 = 9$ lies between a=1.0 and b=2.0. Use the method of bisection four times to find a better approximation of this root. (Note: Carry seven significant digits in your calculations).

(b)Use the following iterative formula once to find a better approximation of the root of the equation given in (a). Take as an initial value the final result you obtained in (a). (Note: Carry seven significant digits in your calculations).

$$x_{n+1} = x_n - \frac{f(x_n)}{f^{(1)}(x_n) - \dfrac{f(x_n)f^{(2)}(x_n)}{2f^{(1)}(x_n)}}$$

[Hint: Let $f(x) = 3^x + x^2 - 9$. Note that $f^{(1)}(x)$ represents the first derivative of f(x). Similarly $f^{(2)}(x)$ represents the second derivative of f(x)].

6.(B) Consider the equation $x^3 - 6x^2 + 9x - 3 = 0$ .This equation can be transformed into the form $x = F(x)$ in several ways. Find a suitable form and use fixed-point iteration six times to find a better approximation to the root that is close to $x_0 = 1.6$.(Note: Carry seven significant digits in your calculations).

7.The matrix $A = \begin{bmatrix} 9 & -6 & 12 \\ -6 & 5 & -3 \\ 12 & -3 & 45 \end{bmatrix}$ can be written as the product of an upper triangular matrix $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$ and its transpose L^T, that is $A = LL^T$.

(a) Find L and L^T.

(b) Use L and L^T to solve the following system of three linear equations:

$$\begin{aligned} 9x - 6y + 12z &= 8 \\ -6x + 5y - 3z &= -8 \\ 12x - 3y + 45z &= -4 \end{aligned}$$

Note: Candidates who use any method other than the one asked for will not get any credit.