National Examination – Dec 2015
04-BS-16: Discrete Mathematics
Duration: 3 hours

Examination Type: Closed Book.
No aids allowed.

# 1:	_______/ 10
# 2:	_______/ 10
# 3:	_______/ 10
# 4:	_______/ 10
# 5:	_______/ 10
# 6:	_______/ 10
# 7:	_______/ 10
# 8:	_______/ 10
# 9:	_______/ 10
# 10:	_______/ 10
# 11:	_______/ 10
# 12:	_______/ 10
TOTAL:	_______/100

This exam paper contains 13 pages (including this one).
Answer 10 out of 12 questions. Ten questions constitute a full paper.
Please clearly indicate which two questions you don't want marked by drawing a diagonal line across the page.
In case of doubt to any question, clearly state any assumptions made.
One of two calculators is permitted any Casio or Sharp approved models.

Good Luck!

Question 1. [10 MARKS]

Part (a) [2 MARKS]

Rewrite the following without negation on qualifiers $\neg\exists x \neg\forall y \neg\exists z P(x, y, z)$

Part (b) [2 MARKS]

Write the sentence "A necessary condition for $P(x, y)$ to be true is that $x > y$" as a logic expression.

Part (c) [3 MARKS]

Is $\exists x \forall y P(x, y) \rightarrow \forall x \exists y P(x, y)$ a tautology? Please either provide a proof or give a counterexample.

Part (d) [3 MARKS]

Consider the universe of discourse as positive intergers. Let $P_n(x, y, z)$ stand for $x^n + y^n = z^n$. Write the Fermat's Last Theorem as a logical proposition, i.e., the equation $x^n + y^n = z^n$ does not have positive integer solution for $n > 2$.

Question 2. [10 MARKS]

Part (a) [5 MARKS]

Show that

$$\sum_{s_1=0}^{1}\sum_{s_2=0}^{1}\cdots\sum_{s_n=0}^{1}\frac{1}{1^{s_1}2^{s_2}\ldots n^{s_n}} = n+1$$

Part (b) [5 MARKS]

Show that the sum of even numbers from $0, 2, \cdots$ to $2n$ is $n(n+1)$.

Question 3. [10 MARKS]

Consider a sequence recursively defined as follows: $a_0 = 2$, $a_{n+1} = a_n^2$.

Part (a) [2 MARKS]

Write down a closed-form expression for a_n.

Part (b) [3 MARKS]

Is $a_n = O(2^n)$? Is $a_n = O(n^n)$?

Part (c) [5 MARKS]

Prove that $a_n - 1$ has at least n distinct prime divisors.

Question 4. [10 MARKS]

A 5-card poker hand is dealt from a 52-card deck. Find the probability of getting

a. Five cards of consecutive rank (2 is the smallest rank, A largest).

b. There is at least one card of each suite.

c. All five cards come from the same suite.

d. There is exactly one pair.

e. Full house: three cards of same rank, plus a pair of different rank.

Question 5. [10 MARKS]

In the world series, two teams play a sequence of up to 7 games. The first team that wins 4 games wins the series. Assume that the teams are evenly matched.

Part (a) [2 MARKS]

What is the probability that the series ends after 4 games?

Part (b) [3 MARKS]

What is the probability that the series ends after the 5th game?

Part (c) [3 MARKS]

What is the probability that the series ends after the 6th game?

Part (d) [2 MARKS]

What is the probability that the series goes to the 7th game?

Question 6. [10 MARKS]

Part (a) [6 MARKS]

Suppose that we have 6 men and 4 women. How many different ways that

a. They can sit in a circular table so that all women sit next to each other? (clockwise and counter-clockwise seatings are regarded as different)

b. A committee of 5 people can be formed so that at most one of John, Mary and Susan is on the committee?

c. A committee of 5 people can be formed with more women than men?

Part (b) [4 MARKS]

How many ways there are to re-arrange the letters in SCIENCE, if

a. there are no restrictions?

b. the C's are together

Question 7. [10 MARKS]

Part (a) [4 MARKS]

Consider the relation R defined on real numbers, where $(a, b) \in R$ if and only if $a - b$ is an integer. Show that R is an equivalence relation. Describe the equivalence classes.

Part (b) [6 MARKS]

Plot the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \sin(x) + x$ over $x \in [-10, 10]$. Is this function one-to-one? onto? Does it have an inverse? If not, specify the largest sets $\mathcal{X}$ and $\mathcal{Y}$ for which the function $f : \mathcal{X} \to \mathcal{Y}$ has an inverse.

Question 8. [10 MARKS]

Part (a) [6 MARKS]

Show that a Fibonacci sequence with the initial condition $a_0 = 0$, $a_1 = 1$, and $a_n = a_{n-1} + a_{n-2}$ can be written in closed-form as

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Part (b) [4 MARKS]

Prove that $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

Question 9. [10 MARKS]

Part (a) [2 MARKS]

Provide a definition of what it means by $f(n)$ is $O(g(n))$?

Part (b) [4 MARKS]

Insertion sort builds a sorted list by inserting one item to the list at a time. Describe how the algorithm works. What is the best-case, the worst-case, and the average run-time complexity of insertion sort? Please explain and provide adequate justification.

Part (c) [1 MARK]

Write down the name of a sorting algorithm that has better average run-time complexity than insertion sort.

Part (d) [3 MARKS]

Please order the following run-time complexity in big-O notation from slowest to fastest.

$$O(n^2),\ O(n\sqrt{n}),\ O(\log(n)),\ O((\log(n))^2),\ O(\log(\log(n))),\ O(2^n),\ O(n^2\log(n)),\ O(1)$$

Question 10. [10 MARKS]

Part (a) [2 MARKS]

Let G be a connected planar simple graph with e edges, and v vertices. Let f be the number of regions in the planar representation of G (including the outer region). What is the relation between e, f and v?

Part (b) [2 MARKS]

A truncated tetrahedron has 4 hexagonal faces and 4 triangle faces. How many vertices and how many edges does it have?

Part (c) [3 MARKS]

Suppose that you use 20 equilateral triangles of same size as faces to construct a polyhedron, you will get a regular icosahedron. How many triangles meet around each vertex?

Part (d) [3 MARKS]

A truncated rhombic dodecahedron consists of square faces and hexagon faces. It has 48 edges and 32 vertices. How many faces are squares and how many hexagons?

Question 11. [10 MARKS]

Part (a) [2 MARKS]

What is an Euler circuit of a graph? Under what condition does a graph have a Euler circuit?

Part (b) [3 MARKS]

For what values of (m, n) does $K_{m,n}$, the complete bipartite graph with m vertices on one side and n vertices on the other, have a Euler circuit? Explain.

Part (c) [2 MARKS]

What is a Hamilton path of a graph?

Part (d) [3 MARKS]

Illustrate whether tetrahedron (four triangle faces), cube (six square faces), and octahedron (eight triangle faces) have a Hamilton path or not.

Question 12. [10 MARKS]

In some cultures, families prefer boys to girls. Suppose that in a society all families keep having more children until a boy is born (and they stop having children as soon as a boy is born). Assume that boys and girls are born with equal probability.

Part (a) [3 MARKS]

Give an expression for the average number of children per family in this society.

Part (b) [2 MARKS]

Give an expression for the average number of girls per family in this society.

Part (c) [1 MARK]

What is the average number of boys per family in this society?

Part (d) [3 MARKS]

Would this society have an imbalance between males and females in the population over the long run? Please explain why or why not.

Total Marks = 100