NATIONAL EXAMINATIONS MAY 2015

04-BS-5 ADVANCED MATHEMATICS

3 Hours duration

NOTES:

1. If doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper, a clear statement of any assumption made.
2. Candidates may use one of the approved Casio or Sharp calculators. This is a Closed Book Exam. However, candidates are permitted to bring **ONE** aid sheet (8.5"x11") written on both sides.
3. Any five (5) questions constitute a complete paper. Only the first five answers as they appear in your answer book will be marked.
4. All questions are of equal value.

Marking Scheme

1. 20 marks
2. (a) 15 marks ; (b) 5 marks
3. (a) 5 marks ; (b) 10 marks ; (c) 5 marks
4. (a) 12 marks ; (b) 8 marks
5. 20 marks
6. (a) 8 marks ; (b) 12 marks
7. (a) 10 marks ; (b) 10 marks

1.Consider the following differential equation:

$$(x^2+2)\frac{d^2y}{dx^2}+3x\frac{dy}{dx}-y=0$$

Find two linearly independent power series solutions about the ordinary point x=0.

2. (a) Find the Fourier series expansion of the periodic function F(x) of period p=2.

$$F(x)=x^2 \; ; \qquad 0 < x < 2$$

(b) Use the result obtained in (a) to prove that

$$\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

3. Consider the following function where a is a positive constant

$$f(x)=\begin{cases}\frac{1}{a}(1+\frac{x}{a}) & -a \le x < 0\\ \frac{1}{a}(1-\frac{x}{a}) & 0 \le x \le a\end{cases}$$

Note that f(x) = 0 for all the other values of x.

(a)Compute the area bounded by f(x) and the x-axis. Graph f(x) against x for $a = 1.0$ and $a = 0.5$.

(b)Find the Fourier transform F(ω) of f(x)

(c)Graph F(ω) against ω for the same two values of a mentioned in (a).
Explain what happens to f(x) and F(ω) when a tends to zero.

Note: $$F(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\exp(-i\omega x)dx$$

4.(A) Prove that the coefficients α and β of the least-squares parabola $Y = \alpha X + \beta X^2$ that fits the set of n points (X_i, Y_i) can be obtained as follows

$$\alpha = \frac{\left\{\sum_{i=1}^{i=n} X_i Y_i\right\}\left\{\sum_{i=1}^{i=n} X_i^4\right\} - \left\{\sum_{i=1}^{i=n} X_i^2 Y_i\right\}\left\{\sum_{i=1}^{i=n} X_i^3\right\}}{\left\{\sum_{i=1}^{i=n} X_i^2\right\}\left\{\sum_{i=1}^{i=n} X_i^4\right\} - \left\{\sum_{i=1}^{i=n} X_i^3\right\}^2};$$

$$\beta = \frac{\left\{\sum_{i=1}^{i=n} X_i^2\right\}\left\{\sum_{i=1}^{i=n} X_i^2 Y_i\right\} - \left\{\sum_{i=1}^{i=n} X_i^3\right\}\left\{\sum_{i=1}^{i=n} X_i Y_i\right\}}{\left\{\sum_{i=1}^{i=n} X_i^2\right\}\left\{\sum_{i=1}^{i=n} X_i^4\right\} - \left\{\sum_{i=1}^{i=n} X_i^3\right\}^2}$$

4.(B) It has been suggested that the following set of n=7 points (X_i, Y_i) are related by an equation of the form $Y = \alpha + \beta X$. Use your calculator to find the least squares estimate of the coefficients α and β.

X	1	3	5	7	9	11
Y	66	52	49	35	23	18

5.The following results were obtained in a certain experiment.

x	0	1	2	3	4	5	6	7	8
f(x)	5	6	8	15	25	36	49	65	83

Use Romberg's algorithm to find an approximate value of the area bounded by the unknown function represented by the table and the lines x=0, x=8 and the x-axis.

Hint: The Romberg algorithm produces a triangular array of numbers, all of which are numerical estimates of the definite integral $\int_a^b f(x)dx$. The array is denoted by the following notation:

$$\begin{array}{llll} R(1,1) & & & \\ R(2,1) & R(2,2) & & \\ R(3,1) & R(3,2) & R(3,3) & \\ R(4,1) & R(4,2) & R(4,3) & R(4,4) \end{array}$$

where

$$R(1,1) = \frac{H_1}{2}\left[f(a) + f(b)\right]$$

$$R(k,,1) = \frac{1}{2}\left[R(k-1,1) + H_{k-1}\sum_{n=1}^{n=2^{k-2}} f(a + (2n-1)H_k)\right]; \qquad H_k = \frac{b-a}{2^{k-1}}$$

$$R(k,j) = R(k,j-1) + \frac{R(k,j-1) - R(k-1,j-1)}{4^{j-1} - 1}$$

6.(a) One root of the equation $6^x - 30x + 10 = 0$ lies between a=2.0. and b=3.0. Use the method of bisection three times to find a better approximation of this root. (Note: Carry five significant digits in your calculations).

6.(b) Use the following iterative formula twice to find a better approximation of the root of the equation given in (a). Take as an initial value the final result you obtained in (a). (Note: Carry seven significant digits in your calculations).

$$x_{n+1} = x_n - \frac{f(x_n)}{f^{(1)}(x_n) - \dfrac{f(x_n)f^{(2)}(x_n)}{2f^{(1)}(x_n)}}$$

[Hint: Let $f(x) = 6^x - 30x + 10$. Note that $f^{(1)}(x)$ represents the first derivative of f(x). Similarly $f^{(2)}(x)$ represents the second derivative of f(x)].

7. The symmetric positive definite matrix $A = \begin{bmatrix} 16 & -8 & -4 \\ -8 & 29 & 12 \\ -4 & 12 & 41 \end{bmatrix}$ can be written as the product of a lower triangular matrix $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$ and its transpose L^T, that is $A=LL^T$.

(a) Find L and L^T.
(b) Use L and L^T to solve the following system of three linear equations:

$$\begin{aligned} 16x - 8y - 4z &= -20 \\ -8x + 29y + 12z &= 80 \\ -4x + 12y + 41z &= 20 \end{aligned}$$