NATIONAL EXAMINATIONS DECEMBER 2015

04-BS-5 ADVANCED MATHEMATICS

3 Hours duration

NOTES:

1. If doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper a clear statement of any assumption made.

2. Candidates may use one of the approved Casio or Sharp calculators. This is a Closed Book Exam. However, candidates are permitted to bring **ONE** aid sheet (8.5"x11") written on both sides.

3. Any five (5) questions constitute a complete paper. Only the first five answers as they appear in your answer book will be marked.

4. All questions are of equal value.

Marking Scheme

1. 20 marks
2. (A) 14 marks ; (B) 6 marks
3. (a) 5 marks ; (b) 9 marks ; (c) 3 marks ; (d) 3 marks
4. (A) 10 marks ; (B) 10 marks
5. 20 marks
6. (a) 7 marks ; (b) 7 marks ; (c) 6 marks
7. (a) 10 marks ; (b) 10 marks

1. Find the eigenvalues and eigenfunctions of the following Sturm-Liouville problem:

$$\frac{d}{dx}(x^{-3}\frac{dy}{dx})+(\lambda+4)x^{-5}y=0 \quad ; \quad y(1)=0 \;;\; y(e^2)=0$$

2.(A) Find the Fourier series expansion of the following periodic function $f(x)$ of period $p=2\pi$.

$$f(x)=\begin{cases} x+\pi & -\pi<x\le 0 \\ \pi & 0<x\le\pi \end{cases}$$

2.(B) By letting $x==0$ in the result obtained in (A) prove that

$$\frac{\pi^2}{8}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

3.Consider the following function where $K$ and $a$ are positive constants

$$\mathrm{f(x)}=\begin{cases} \frac{K}{2a}\exp(\frac{x}{a}) & x<0 \\ \frac{K}{2a}\exp(-\frac{x}{a}) & x>0 \end{cases}$$

(a) Compute the area bounded by f(x) and the x-axis. Graph f(x) against x for $K=20$, $a=2$ and $a=1$.
(b) Find the Fourier transform F(ω) of f(x).
(c) Graph F(ω) against ω for the values of $K$ and $a$ mentioned in (a).
(d) Explain what happens to f(x) and F(ω) when $a$ tends to zero.

Note: $$\mathrm{F}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{f(x)}\exp(-i\omega x)dx$$

4(A) Set up Newton's divided difference formula for the data tabulated below and derive the polynomial of highest possible degree.

| x | –3 | –2 | 0 | 3 | 5 | 6 |
|---|---|---|---|---|---|---|
| F(x) | –28 | 0 | 8 | –10 | 28 | 80 |

4(B) Find the Lagrange polynomial that fits the following set of four points.

| x | −5 | −2 | 0 | 4 |
|---|---|---|---|---|
| F(x) | −162 | 0 | 8 | 0 |

5.The following results were obtained in a certain experiment.

| x | −4 | −3 | −2 | −1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| y | 113 | 116 | 118 | 119 | 124 | 139 | 164 | 198 | 239 |

Use Romberg's algorithm to obtain an approximate value of the area bounded by the unknown curve represented by the table and the lines x = −4, x = 4 and the x-axis.

Note: The Romberg algorithm produces a triangular array of numbers, all of which are numerical estimates of the definite integral $\int_a^b f(x)dx$. The array is denoted by the following notation:

| | | | |
|---|---|---|---|
| R(1,1) | | | |
| R(2,1) | R(2,2) | | |
| R(3,1) | R(3,2) | R(3,3) | |
| R(4,1) | R(4,2) | R(4,3) | R(4,4) |

where

$$R(1,1) = \frac{H_1}{2}[f(a) + f(b)]$$

$$R(k,1) = \frac{1}{2}\left[R(k-1,1) + H_{k-1}\sum_{n=1}^{n=2^{k-2}} f(a+(2n-1)H_k)\right]; \qquad H_k = \frac{b-a}{2^{k-1}}$$

$$R(k,j) = R(k,j-1) + \frac{R(k,j-1) - R(k-1,j-1)}{4^{j-1}-1}$$

6.(A)The equation $\ln(x+2) - x^2 + 6x - 5 = 0$ has a root in the neighbourhood of $x_0 = 5.0$. Use Newton's method three times to find a better approximation of the root. (Note: Carry at least seven digits in your calculations)

6.(B The equation given in (A) can be written in the form $x = g(x)$ in several obvious ways. One way is to write it in the form

$$x = \{\ln(x+2) - 5\}/(x - 6)$$

Apply the method of fixed-point iteration six times to find the root that is close to $x_0 = 1$ using this form. Explain clearly why this form converges to the root. (Note: Carry seven digits in your computations).

6.(C) The positive root of the equation $2\cos(x/2) - x - 1 = 0$ lies between $\alpha = 0.82$ and $\beta = 0.84$. Use the method of bisection three times to find a better approximation of this root.

7.The matrix $A = \begin{bmatrix} 6 & -1 & 5 \\ -18 & 7 & -13 \\ 12 & 18 & 21 \end{bmatrix}$ can be written as the product of a lower triangular matrix $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and an upper triangular matrix $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$, that is $A=LU$.

(a) Find L and U.

(b) Use the results obtained in (a) to solve the following system of three linear equations:

$$\begin{aligned} 6x_1 - x_2 + 5x_3 &= 8 \\ -18x_1 + 7x_2 - 13x_3 &= -30 \\ 12x_1 + 18x_2 + 21x_3 &= -13 \end{aligned}$$