National Exams May 2014

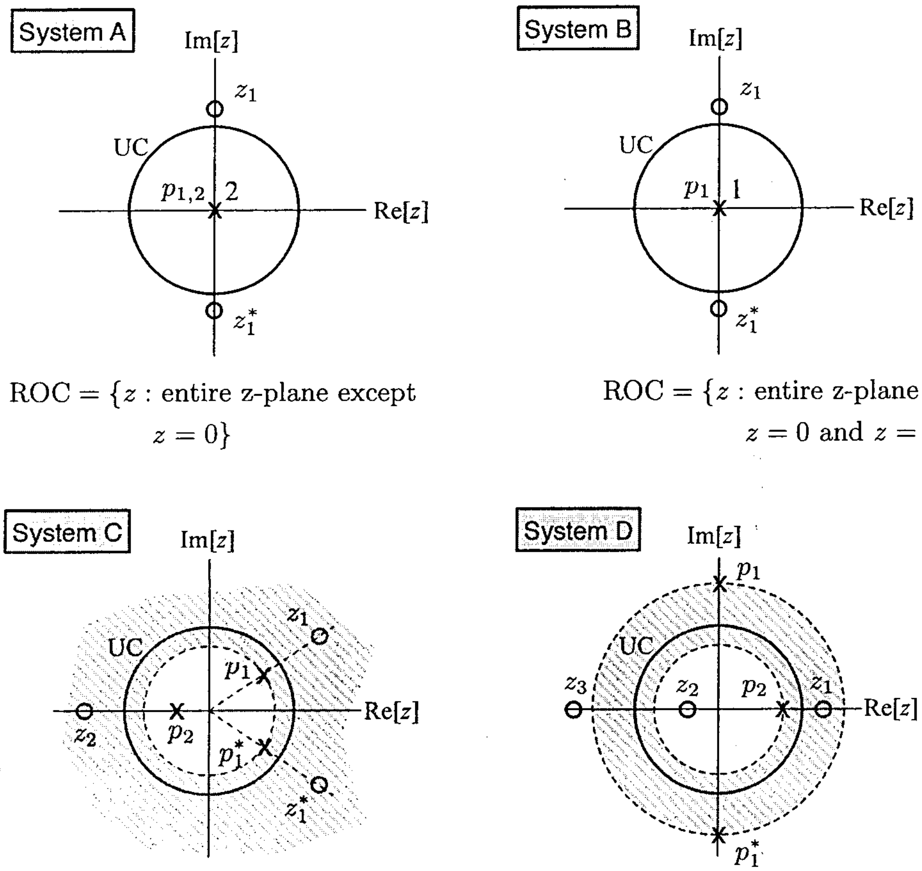
07-Elec-B1, Digital Signal Processing

3 hours duration

NOTES:

1. If doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper, a clear statement of any assumptions made.
2. This is an OPEN BOOK EXAM. Any non-communicating calculator is permitted.
3. **Five (5)** questions constitute a complete paper. The first five questions as they appear in your answer book will be marked.
4. All questions are of equal value.
5. You can find a list of commonly used symbols, some trigonometric identities, Discrete-Fourier Transform definitions and formulations, Discrete-Time Fourier Transform tables and z-Transform tables at the end of this exam paper.

1. Figure (1) depicts the pole-zero diagrams corresponding to the transfer functions of four independent systems. ***Note:*** *In all diagrams* ***UC*** *represents the unit circle. The region-of-convergence of each system is indicated below the corresponding pole-zero diagram.*



System A: ROC = $\{z$: entire z-plane except $z = 0\}$

System B: ROC = $\{z$: entire z-plane except $z = 0$ and $z = \infty\}$

System C: ROC $= \{z : |z| > |p_1|\}$

System D: ROC $= \{z : |p_2| < |z| < |p_1|\}$

Figure 1: Pole-zero diagrams of four different systems.

(a) For each system determine the following system characteristics:

- Type of system (FIR or IIR);
- Stability;
- Causality.

Explain and justify your answers. Answers without explanations will not receive any credit.

(b) For each system determine the whether the corresponding unit impulse response sequence $h[n]$ is a double-sided, left-sided, right-sided, mixed or finite length sequence.

2. Let $x(t)$ be a real-valued analog signal applied to the input of the A/D which operates at the sampling rate $F_s = 48$ kHz. The input signal $x(t)$ is corrupted by two strong real-valued sinusoids with frequencies F_1 and F_2, such that $0 < F_1 < F_2 < 24$ kHz. Figure (2) displays the magnitude spectrum of the 64-point DFT obtained from the 48 kHz sampled sequence $x[n]$.

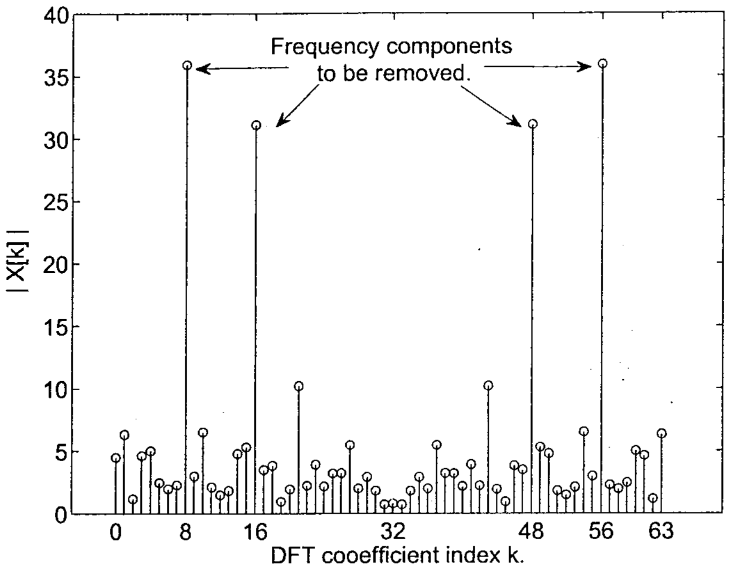


Figure 2: Magnitude spectrum of 64-point DFT of $x[n]$.

(a) From the magnitude spectrum shown in Figure (2) identify the frequencies F_1 and F_2 of the two real-valued sinusoids.

(b) Design a causal FIR notch filter with real-valued coefficients which would remove the unwanted sinusoids at frequencies F_1 and F_2. Determine the system function, $H_{notch}(z)$, of the filter.

(c) Sketch the pole-zero diagram of $H_{notch}(z)$. Sketch a signal-flow/block diagram which implements the notch filter described by $H_{notch}(z)$ using a minimum number of multiplications. Determine the number of additions/subtractions, multiplications and storage requirements for this implementation.

(d) If the input to $H_{notch}(z)$ has been sampled at 10 kHz, which frequencies will be blocked by the filter?

(e) Convert the FIR filter $H_{notch}(z)$ into an IIR filter for improved performance. The IIR filter should again remove the *d.c.* component and the sinusoids at frequencies F_1 and F_2, while reducing the bandwidth of the notches in the magnitude response of the filter. Determine the transfer function of this IIR filter and provide a signal-flow/block diagram representation.

3. Let $\mathbf{x} = \{x[0], \ldots, x[N-1]\}$ be an N-element sequence. Let $\mathbf{X} = \{X[0], \ldots, X[N-1]\}$ be the corresponding N-point DFT of $\mathbf{x}$ such that

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn}, \qquad k = 0, 1, \ldots, N-1; \tag{1}$$

where $W_N = e^{-j2\pi/N}$. An engineer wants to recover $\mathbf{x}$ by applying the inverse transform IDFT to $\mathbf{X}$. However, as a result of carelessness the engineer applies DFT to $\mathbf{X}$ again and generates the sequence $\mathbf{y} = \{y[0], \ldots, y[N-1]\}$:

$$y[m] = \sum_{k=0}^{N-1} X[k]\, W_N^{mk}, \qquad m = 0, 1, \ldots, N-1; \tag{2}$$

Your task is to help the engineer by exploring if it is still possible to recover $\mathbf{x}$ from $\mathbf{y}$.

(a) If $\mathbf{x} = \{0, 1, -1, 2\}$, determine $\mathbf{y}$. Does it appear that $\mathbf{x}$ can be recovered from $\mathbf{y}$?

(b) Prove that $\mathbf{x}$ can be recovered from $\mathbf{y}$ for the general case when $\mathbf{x}$ is an arbitrary N-element sequence. If $X[k]$ and $y[m]$ are given by Equations (1) and (2), respectively, determine $\mathbf{y}$ in terms of $\mathbf{x}$. **Hint:** Consider using the relation:

$$\frac{1}{N}\sum_{k=0}^{N-1} W_N^{(n-l)k} = \begin{cases} 1, & n = \langle l \rangle_N; \\ 0, & otherwise. \end{cases} \tag{3}$$

to simplify the expressions, where the notation $\langle\, . \,\rangle$ denotes the *modulo* operation:

$$\langle l \rangle_N = l \text{ modulo } N \tag{4}$$

(c) Using the relation you derived in part (b), show how the elements of $\mathbf{y}$ are related to the elements of $\mathbf{x} = \{x[0], x[1], \ldots, x[15]\}$.

(d) If the DFT operations used in transforming $\mathbf{x}$ into $\mathbf{X}$, and $\mathbf{X}$ into $\mathbf{y}$ are implemented **directly** using Equations (1) and (2), respectively, determine the total number of real multiplications needed to generate $\mathbf{y}$ from $\mathbf{x}$. Also determine the total number of real multiplications needed if the DFTs are implemented using a radix-2 FFT algorithm. Assume $N = 2^q$ with $q \in \mathcal{Z}^+$.

4. Consider the discrete-time, linear, time-invariant system $\mathcal{S}$ described by the difference equation:

$$y[n] = x[n] - x[n-1]. \tag{5}$$

Let the input to the system be the 3-sample sequence

$$x[n] = \{ \underset{\underset{n=0}{\uparrow}}{1}, \quad 0, \quad -1 \} \tag{6}$$

(a) Determine the unit impulse response sequence $h[n]$ that describes the system $\mathcal{S}$.

(b) Determine the system output $y[n]$ when $x[n]$ is the input using the **iterative** solution of the difference equation given in Equation (5).

(c) Determine $y[n]$ using **linear convolution**.

(d) Determine $y[n]$ using **circular convolution** (implemented in time domain).

(e) Determine $y[n]$ using **DFT/IDFT** operations.

5. Let S be a discrete-time, linear, time-invariant system which is known to be **causal** and **stable**. The system S is described by the transfer function $H(z)$ with the following characteristics:

   - System Zeros: $z_1 = 0$ and $z_2 = -1$;
   - System Poles: $p_1 = 0.5$ and $p_2 = -0.5$;
   - $H(1) =$ d.c. gain $= 8/3$.

   (a) Determine the **transfer function** $H(z)$ and its **region-of-convergence**.

   (b) Determine the **difference equation** that describes the system.

   (c) Determine the **unit impulse response sequence** $h[n]$.

   (d) Let $x[n] = (\frac{1}{3})^n u[n]$ be the input to the system where $u[n]$ is the unit step sequence. The initial conditions for the system are $y[-1] = 0$ and $y[-2] = 8$. Determine the **total system response** $y[n]$. You may use any method of your choice.

   (e) Sketch a **canonic realization** of the system that uses minimum number of delay elements. Determine the number of additions, multiplications and delay elements used by your realization.

6. Let $\mathcal{S}$ be a discrete-time, linear, time-invariant system which is known to be **causal** and **stable**. The system $\mathcal{S}$ is described by the difference equation:

$$y[n] = \left(\frac{1}{2}\right)y[n-1] + x[n] + \left(\frac{1}{2}\right)x[n-1]. \tag{7}$$

   (a) Determine the impulse (unit sample) response sequence $h[n]$.

   (b) Determine the frequency response function $H(e^{j\omega})$.

   (c) Determine the response of the system to the input $x[n] = \cos\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)$.

7. In this question we investigate how we can use the DFT algorithm to calculate the inverse-DFT (IDFT). The box labeled "N-point DFT" shown in Figure (3), operates on the input sequence $\{x[n]\}_{n=0}^{N-1}$ and generates the N-point DFT sequence $\{X[k]\}_{k=0}^{N-1}$, where $\mathbf{Re}[z]$ and $\mathbf{Im}[z]$ refer to the *real* and *imaginary* parts of z, respectively.

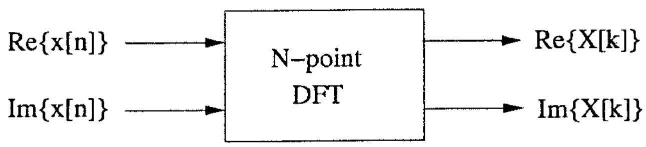


Figure 3: Representation of the N-point DFT block.

(a) Determine the modifications that are **external** to the "box", such that when the input to the modified system is $\{X[k]\}_{k=0}^{N-1}$ then the output of the modified system will be $\{x[n]\}_{n=0}^{N-1}$.

(b) A second approach to the IDFT computation using a DFT algorithm is illustrated in Figure (4).

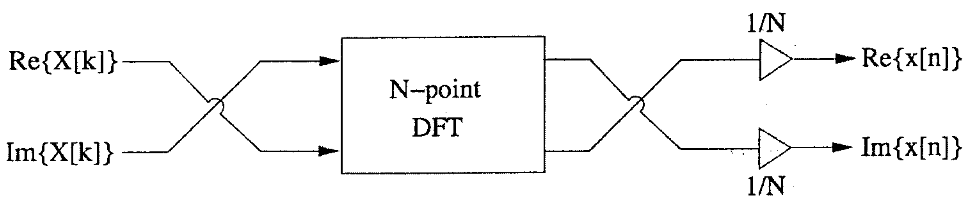


Figure 4: Computing the IDFT from DFT using Equation (9).

Define a length-N sequence $\{q[n]\}_{n=0}^{N-1}$ as

$$\mathbf{Re}[q[n]] = \mathbf{Im}X[k]\Big|_{k=n}, \quad \mathbf{Im}[q[n]] = \mathbf{Re}X[k]\Big|_{k=n}. \tag{8}$$

with $Q[k]$ denoting its N-point DFT. Demonstrate that this approach will indeed allow you to calculate the IDFT by showing that

$$\mathbf{Re}[x[n]] = \frac{1}{N}\mathbf{Im}\,Q[k]\Big|_{k=n}, \qquad \mathbf{Im}[x[n]] = \frac{1}{N}\mathbf{Re}\,Q[k]\Big|_{k=n}. \tag{9}$$

(c) Let $X[k] = \{1, 1+2j, 1, 1-2j\}$ be the 4-point DFT of the time-domain sequence $x[n]$. Determine $x[n]$ by evaluating the IDFT of $X[k]$ using the approach delineated in part (b). (No credit will be given if $x[n]$ is determined using an approach different than the one described in part (b).)

List of Commonly Used Symbols:

$a(t), x(t), y(t), \ldots$	continuous-time signals.
$a[n], x[n], y[n], \ldots$	discrete-time sequences.
$A(z)$	z-Transform of the discrete-time sequence $a[n]$
$h[n]$	impulse response sequence of a discrete-time system.
$H(z)$	transfer function of a discrete-time linear, time-invariant system represented by the impulse response sequence $h[n]$.
Ω	angular frequency.
F	frequency such that $\Omega = 2\pi F$.
ω	digital angular frequency.
f	digital frequency such that $\omega = 2\pi f$.
$X(\omega)$	Discrete-Time Fourier Transform of $x[n]$ defined as $X(\omega) = \sum_n x[n]e^{-j\omega n}$.
$X(\Omega)$	Continuous-Time Fourier Transform of $x(t)$ defined as $X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$.

Trigonometric Identities:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin\alpha\ \sin\beta = \tfrac{1}{2}\cos(\alpha-\beta) - \tfrac{1}{2}\cos(\alpha+\beta)$$
$$\cos\alpha\ \cos\beta = \tfrac{1}{2}\cos(\alpha+\beta) + \tfrac{1}{2}\cos(\alpha-\beta)$$
$$\sin\alpha\ \cos\beta = \tfrac{1}{2}\sin(\alpha+\beta) + \tfrac{1}{2}\sin(\alpha-\beta)$$

Euler's Identity:

$$e^{j\theta} = \cos\theta\ +\ j\ \sin\theta$$

Discrete Fourier Transform: The N-point DFT of a N-sample sequence $s[n]$ is defined as:

$$S[k] = \sum_{n=0}^{N-1} s[n]\, W_N^{kn}, \qquad k = 0, 1, \ldots, N-1.$$

The sequence $s[n]$ can be recovered from its DFT coefficients using the N-point IDFT:

$$s[n] = \frac{1}{N}\sum_{k=0}^{N-1} S[k]\, W_N^{-kn}, \qquad n = 0, 1, \ldots, N-1.$$

where $W_N = e^{-j2\pi/N}$. The DFT/IDFT relations can also be expressed in matrix form

$$\begin{bmatrix} S[0] \\ \vdots \\ S[N-1] \end{bmatrix} = \mathbf{W}_N \begin{bmatrix} s[0] \\ \vdots \\ s[N-1] \end{bmatrix} \qquad \begin{bmatrix} s[0] \\ \vdots \\ s[N-1] \end{bmatrix} = \mathbf{W}_N^{-1} \begin{bmatrix} S[0] \\ \vdots \\ S[N-1] \end{bmatrix}$$

where the transformation matrices for $N = 2, 3, 4$ are

$$\mathbf{W}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \mathbf{W}_2^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\mathbf{W}_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2}-\frac{\sqrt{3}}{2}j & -\frac{1}{2}+\frac{\sqrt{3}}{2}j \\ 1 & -\frac{1}{2}+\frac{\sqrt{3}}{2}j & -\frac{1}{2}-\frac{\sqrt{3}}{2}j \end{bmatrix} \qquad \mathbf{W}_3^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2}+\frac{\sqrt{3}}{2}j & -\frac{1}{2}-\frac{\sqrt{3}}{2}j \\ 1 & -\frac{1}{2}-\frac{\sqrt{3}}{2}j & -\frac{1}{2}+\frac{\sqrt{3}}{2}j \end{bmatrix}$$

$$\mathbf{W}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \qquad \mathbf{W}_4^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & +j \end{bmatrix}$$

Table 1: Symmetry Properties of the Discrete-Time Fourier Transform

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{R}e\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
4. $j\mathcal{I}m\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)	$X_R(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)	$jX_I(e^{j\omega}) = j\mathcal{I}m\{X(e^{j\omega})\}$
The following properties apply only when $x[n]$ is real:	
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ (magnitude is even)
11. Any real $x[n]$	$\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$)	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$)	$jX_I(e^{j\omega})$

Table 2: Discrete-Time Fourier Transform Theorems

Sequence	Fourier Transform
$x[n]$	$X(e^{j\omega})$
$y[n]$	$Y(e^{j\omega})$
1. $ax[n]+by[n]$	$aX(e^{j\omega})+bY(e^{j\omega})$
2. $x[n-n_d]$ (n_d an integer)	$e^{-j\omega n_d}X(e^{j\omega})$
3. $e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
6. $x[n]*y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Parseval's theorem:

8. $$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

9. $$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

Table 3: Discrete-Time Fourier Transform Pairs

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n-n_0]$	$e^{-j\omega n_0}$
3. $1 \quad (-\infty < n < \infty)$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
4. $a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{1-ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
6. $(n+1)a^n u[n] \quad (\|a\| < 1)$	$\frac{1}{(1-ae^{-j\omega})^2}$
7. $\frac{r^n \sin\omega_p(n+1)}{\sin\omega_p} u[n] \quad (\|r\| < 1)$	$\frac{1}{1-2r\cos\omega_p e^{-j\omega} + r^2 e^{-j2\omega}}$
8. $\frac{\sin\omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, & \|\omega\| < \omega_c, \\ 0, & \omega_c < \|\omega\| \le \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$
11. $\cos(\omega_0 n + \phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi}\delta(\omega - \omega_0 + 2\pi k) + \pi e^{-j\phi}\delta(\omega + \omega_0 + 2\pi k)]$

Table 4: Some Common z-Transform Pairs

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
9. $\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$

Table 5: Some z-Transform Properties

Sequence	Transform	ROC
$x[n]$	$X(z)$	R_x
$x_1[n]$	$X_1(z)$	R_{x_1}
$x_2[n]$	$X_2(z)$	R_{x_2}
$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
$x[n - n_0]$	$z^{-n_0} X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\| R_x$
$nx[n]$	$-z\frac{dX(z)}{dz}$	R_x
$x^*[n]$	$X^*(z^*)$	R_x
$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Marking Scheme

Question 1: **(a)** 10, **(b)** 10 marks.

Question 2: **(a)** 3, **(b)** 4, **(c)** 5, **(d)** 4, **(e)** 4 marks.

Question 3: **(a)** 5, **(b)** 5, **(c)** 5, **(d)** 5 marks.

Question 4: **(a)** 2, **(b)** 4, **(c)** 4, **(d)** 4, **(e)** 6 marks.

Question 5: **(a)** 3, **(b)** 3, **(c)** 4, **(d)** 7, **(e)** 3 marks.

Question 6: **(a)** 7, **(b)** 8, **(c)** 5 marks.

Question 7: **(a)** 6, **(b)** 9, **(c)** 5 marks.