National Examination – May 2014
04-BS-16: Discrete Mathematics
Duration: 3 hours

Examination Type: Closed Book.
No aids allowed.

Last Name: _____

First Name: _____

*Do **not** turn this page until you have received the signal to start.*
(In the meantime, please fill out the identification section above, and read the instructions below.)

This exam paper contains 13 pages (including this one).
Answer 10 out of 12 questions. Ten questions constitute a full paper.
Please clearly indicate which two questions you don't want marked by drawing a diagonal line across the page.
In case of doubt to any question, clearly state any assumptions made.

1: _____/ 10
2: _____/ 10
3: _____/ 10
4: _____/ 10
5: _____/ 10
6: _____/ 10
7: _____/ 10
8: _____/ 10
9: _____/ 10
10: _____/ 10
11: _____/ 10
12: _____/ 10

TOTAL: _____/100

Good Luck!

Question 1. [10 MARKS]

Part (a) [6 MARKS]

Let the universe of discourse be positive integer. Let

- $P(x)$: "x is primes"
- $L(x, y)$: "x is less than y"
- $E(x)$: "x is even"
- $A(x, y, z)$: "$x + y = z$"

Write the following statement using predicates, quantifiers and logic connectives.

a. Not all positive integers are prime.

b. There exists an even prime.

c. Every even integer greater than 2 is a sum of two prime numbers.

d. There does not exist the largest prime, (i.e., there are infinitely many primes).

e. There are infinitely many twin primes. (Twin primes are two prime numbers of the form n and $n + 2$.)

Question 2. [10 MARKS]

Part (a) [5 MARKS]

Show that

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Part (b) [5 MARKS]

Show that for all $n \geq 1$, $9^n - 2^n$ is divisible by 7.

Question 3. [10 MARKS]

A 5-card poker hand is dealt from a 52-card deck. Find the probability of getting

a. Four of a kind, (four cards of the same rank).

b. Flush or straight flush, (all five cards are of the same suit, may or may not be in sequence).

c. Two pairs, (two cards of the same rank, plus two cards of another rank, plus another card of different rank).

d. Three of a kind, (three cards of the same rank, plus two cards which are not of this rank nor the same rank as each other).

e. Full house, (three cards of same rank, and two matching cards (i.e., pair) of different rank)

Question 4. [10 MARKS]

Part (a) [6 MARKS]

Among 12 men and 12 women, how many committees can be chosen so that the committee consists of 8 people and

a. has at least one woman and at least one man?

b. with equal number of men and women?

c. cannot have both John and Rob serving on the same committee?

Part (b) [4 MARKS]

How many ways there are to re-arrange the letters in ABBACADABRA, if

a. there are no restrictions?

b. all the A's are together

Question 5. [10 MARKS]

Part (a) [6 MARKS]

Prove

- $p \to (q \to r)$ is equivalent to $(p \wedge q) \to r$.

- $((a \wedge p) \vee p) \to p)$ is a tautology.

Part (b) [4 MARKS]

a. For a sequence a_n, define what it means that $a_n = O(f(n))$.

b. A Fibonacci sequence with the initial condition, $a_1 = 3$, $a_2 = 4$ and $a_n = a_{n-1} + a_{n-2}$ can be written in closed-form as

$$a_n = \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

What is the smallest x for which $a_n = O(x^n)$? Explain.

Question 6. [10 MARKS]

Part (a) [5 MARKS]

The English alphabet contains 26 letters, 5 of which are vowels (a, e, i, o, u), and 21 are consonants. Show that if the 26 letters are arranged in a sequence, there must be at least one occurrence of 4 consecutive consonants in the sequence.

Part (b) [3 MARKS]

What is the probability that if we arrange the letters A, A, A, B, B, B at random, the resulting sequence happens to be exactly AAABBB?

Part (c) [2 MARKS]

How many different functions are there mapping the 26-letter English alphabet to binary numbers $\{0, 1\}$?

Question 7. [10 MARKS]

Part (a) [5 MARKS]

Consider a set of pairs of integers (m, n) with $n \neq 0$. Define a relation as follows: (m, n) and (p, q) are related if $\frac{m}{n} = \frac{p}{q}$. Explain whether this relation is an equivalence relation. If so, describe the set of equivalence classes in the simplest mathematical term.

Part (b) [5 MARKS]

Consider $f : \mathbb{R} \to \mathbb{Z}$, $f(x) = \lfloor x^3 + 0.5 \rfloor$, where $\mathbb{R}$ is the set of real numbers, $\mathbb{Z}$ is the set of integers, and $\lfloor \cdot \rfloor$ is the floor operation. Is f one-to-one? Is f onto? Does f have an inverse? Explain.

Question 8. [10 MARKS]

Part (a) [5 MARKS]

Below are the adjacency matrices of two graphs. Please draw the graphs, and explain whether they are isomorphic.

$$\begin{bmatrix} 0&1&0&1&0&0&0&0 \\ 1&0&1&0&0&1&0&0 \\ 0&1&0&1&0&0&0&0 \\ 1&0&1&0&0&0&0&1 \\ 0&0&0&0&0&1&0&1 \\ 0&1&0&0&1&0&1&0 \\ 0&0&0&0&0&1&0&1 \\ 0&0&0&1&1&0&1&0 \end{bmatrix} \qquad \begin{bmatrix} 0&1&0&1&1&0&0&0 \\ 1&0&1&0&0&0&0&0 \\ 0&1&0&1&0&0&0&0 \\ 1&0&1&0&0&0&0&1 \\ 1&0&0&0&0&1&0&1 \\ 0&0&0&0&1&0&1&0 \\ 0&0&0&0&0&1&0&1 \\ 0&0&0&1&1&0&1&0 \end{bmatrix}$$

Part (b) [5 MARKS]

How many non-isomorphic simple graphs are there with 4 vertices? Please draw them all.

Question 9. [10 MARKS]

Part (a) [7 MARKS]

Describe how quick sort works. What is the average run-time complexity of quick sort?

Part (b) [3 MARKS]

Please order the following run-time complexity in big-O notation from slowest to fastest.

$$O(n^n),\ O(\sqrt{n}),\ O(n^{\log(n)}),\ O(\log(n)),\ O(n\log(n)),\ O(n^4),\ O(n^3\log(n)),\ O(2^{\sqrt{n}})$$

Question 10. [10 MARKS]

Part (a) [2 MARKS]

What is a Euler circuit? Under what condition does a simple connected graph have an Euler circuit?

Part (b) [2 MARKS]

What is a Euler path? Under what condition does a simple connected graph have an Euler path?

Part (c) [2 MARKS]

Which complete graphs K_n have an Euler circuit?

Part (d) [2 MARKS]

Which complete bipartite graphs $K_{m,n}$ have an Euler circuit?

Part (e) [2 MARKS]

Which complete bipartite graphs $K_{m,n}$ have an Euler path?

Question 11. [10 MARKS]

Part (a) [4 MARKS]

It is estimated that in a population of 1000 people in the winter, at any given time, 10 are having a headache, 20 are having fever, and 5 have both.

a. What is the probability that a person is sick (either fever or headache)?

b. What is the probability that a person has headache given that the person has a fever?

c. What is the probability that a person has fever given that the person has headache?

d. What is the probability that a person has headache given that the person does not have fever?

Part (b) [6 MARKS]

A byte consists of 8 bits. Suppose that we generate a byte at random. Each bit has 50% probability of being 1 or 0. What is the probability that it contains no more than 4 consecutive 0's?

Question 12. [10 MARKS]

Part (a) [2 MARKS]

Euler's polyhedron formula relates the number of vertices, faces, and edges in a polydedron. Write down this relationship.

Part (b) [3 MARKS]

In computer graphics, 3-dimensional objects are often represented by triangle meshes. Suppose that a triangle mesh representing a polyhedron consists of K triangles, where each edge is shared by two triangles but each vertex may be shared by multiple triangles. How many vertices are there in such a mesh?

Part (c) [5 MARKS]

Bucky-ball C_{60} (buckminsterfullerene) is a truncated icosahedron with 60 vertices and 32 faces. The faces are either hexagons or pentagons. Among the 32 faces, how many are hexagons and how many pentagons?

Total Marks = 100