National Exams May 2019

16-Elec-B1, Digital Signal Processing

3 hours duration

NOTES:

1. If doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper, a clear statement of any assumptions made.

2. This is a Closed Book exam.
   Candidates may use one of two calculators, the Casio or Sharp approved models. They are also entitled to one aid sheet with tables & formulas written both sides. No textbook excerpts or examples solved.

3. FIVE (5) questions constitute a complete exam.
   Clearly indicate your choice of any five of the six questions given otherwise the first five answers found will be considered your pick.

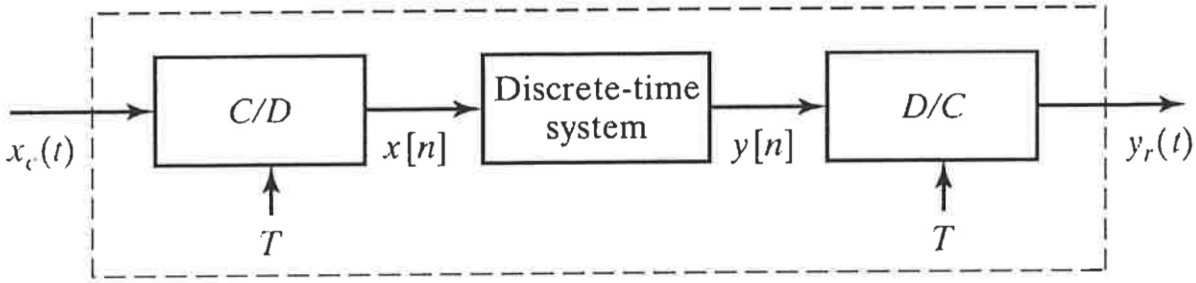
4. All questions are worth 12 points.
   See below for a detailed breakdown of the marking.

Marking Scheme

1. (a) 6, (b) 6, total = 12
2. (a) 6, (b) 6, total = 12
3. (a) 6, (b) 6, total = 12
4. (a) 7, (b) 5, total = 12
5. (a) 3, (b) 2, (c) 2, (d) 3, (e) 2, total = 12
6. (a) 5, (b) 3, (c) 4, total = 12

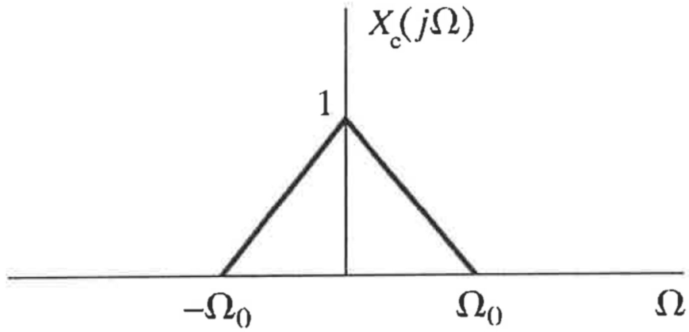
The number beside each part above indicates the points that part is worth

1.- Consider the system in the figure below.



The input signal $x_c(t)$ has the Fourier transform shown in the figure below with $\Omega_0 = 2\pi(1000)$ rad/s. The discrete-time system is an ideal lowpass filter with frequency response

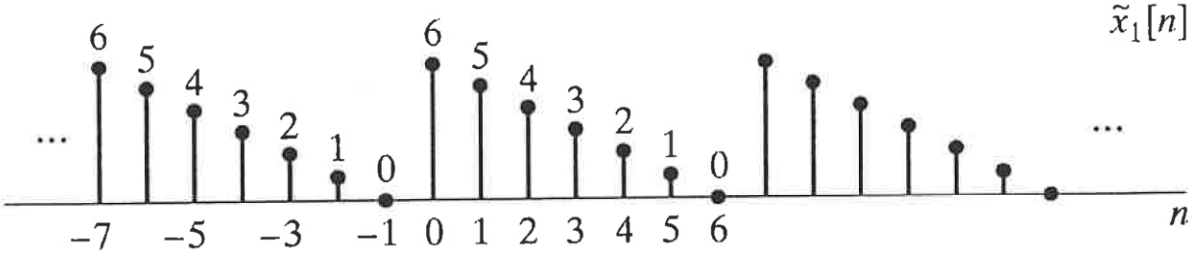
$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c, \\ 0, & otherwise. \end{cases}$$

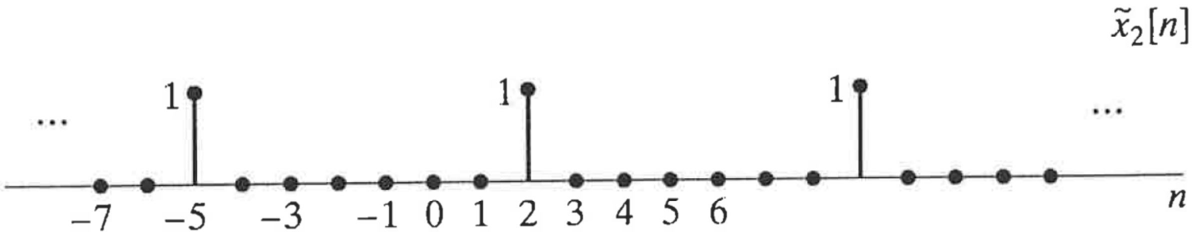


(a) What is the minimum sampling rate $F_s = 1/T$ such that no aliasing occurs in sampling the input?

(b) If $\omega_c = \pi/2$, what is the minimum sampling rate such that $y_r(t) = x_c(t)$?

2.- The figure below shows two periodic sequences, $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$, with period $N = 7$.

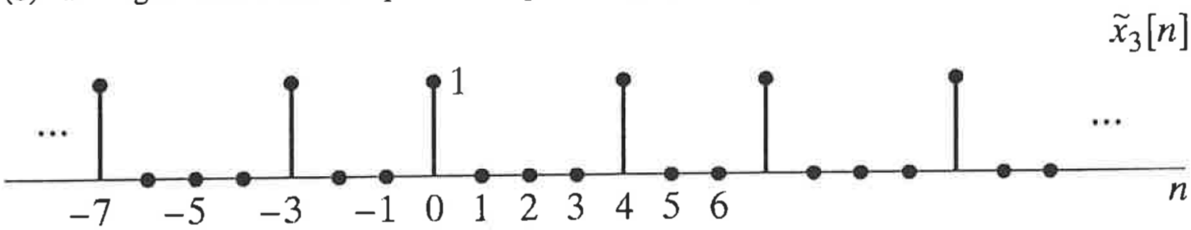




(a) Find a sequence $\tilde{y}_1[n]$ whose DFS is equal to the product of the DFS of $\tilde{x}_1[n]$ and the DFS of $\tilde{x}_2[n]$, *i.e.*,

$$\tilde{Y}_1[k] = \tilde{X}_1[k]\,\tilde{X}_2[k].$$

(b) The figure below shows a periodic sequence $\tilde{x}_3[n]$ with period $N = 7$.



Find a sequence $\tilde{y}_2[n]$ whose DFS is equal to the product of the DFS of $\tilde{x}_1[n]$ and the DFS of $\tilde{x}_3[n]$, *i.e.*,

$$\tilde{Y}_2[k] = \tilde{X}_1[k]\,\tilde{X}_3[k].$$

3.- Suppose you have a signal $x[n]$ with 498 nonzero samples whose discrete-time Fourier transform you wish to estimate by computing the DFT. You find that it takes your computer 1 second to compute the 498-point DFT of $x[n]$. You then add fourteen zero-valued samples at the end of the sequence $x[n]$ to form a 512-point sequence $x_1[n]$. Now the computation of $X_1[k]$ takes your computer just 9.29 milliseconds. Reflecting, you realize that by using $x_1[n]$, you are able to compute more samples of $X(e^{j\omega})$ in a much shorter time by adding some zeros to the end of $x[n]$ and using this slightly longer sequence.

(a) How do you explain this apparent paradox?

(b) Justify the difference in computational time numerically.

4.- (a) A discrete-time LTI system has system function given by

$$H(z) = \frac{1}{1 - \frac{1}{4}z^{-2}}$$

and impulse response given by

$$h[n] = A_1 \propto_1^n u[n] + A_2 \propto_2^n u[n].$$

(i) Determine the values of $A_1, A_2, \propto_1$ and $\propto_2$.

(ii) Is this system stable? Justify your answer.

(b) If the system function is given by

$$H(z) = \frac{1 - \frac{1}{1024}z^{-10}}{1 - \frac{1}{2}z^{-1}}, \quad \text{for } |z| > 0.$$

Is the corresponding LTI system causal? Justify your answer.

5.- Consider a causal LTI system whose system function is

$$H(z) = \frac{1 - \frac{3}{10}z^{-1} + \frac{1}{3}z^{-2}}{\left(1 - \frac{4}{5}z^{-1} + \frac{2}{3}z^{-2}\right)\left(1 + \frac{1}{5}z^{-1}\right)}$$

Draw the signal flow graphs for implementations of the system in each of the following forms:

(a) Direct Form I

(b) Direct Form II

(c) Cascade Form using 1st and 2nd-order direct form II sections

(d) Parallel Form using 1st and 2nd-order direct form I sections

(e) Transposed Direct Form II

6.- We wish to use the Kaiser window method to design a discrete-time filter with linear phase that meets the following specifications:

$$|H(e^{j\omega})| \le 0.01, \quad 0 \le |\omega| \le 0.25\pi,$$
$$0.95 \le |H(e^{j\omega})| \le 1.05, \quad 0.35\pi \le |\omega| \le 0.6\pi,$$
$$|H(e^{j\omega})| \le 0.01, \quad 0.65\pi \le |\omega| \le \pi.$$

(a) Determine the minimum length $(M+1)$ of the impulse response and the value of the Kaiser window parameter β for a filter that meets the preceding specification

(b) What is the delay introduced by the filter?

(c) Determine the ideal impulse response $h_d[n]$ to which the Kaiser window should be applied.

Formulas for Kaiser window parameters:

$$\beta = \begin{cases} 0.1102(A-8.7), & A > 50, \\ 0.5842(A-21)^{0.4} + 0.07886(A-21), & 21 \le A \le 50, \\ 0.0, & A < 21. \end{cases} \quad \textit{where } A = -20\log_{10}\delta,$$

$$M = \frac{A-8}{2.285\Delta\omega} \quad \textit{where } \Delta\omega \textit{ is the transition band width in the design specifications.}$$

Additional Information

(Not all of this information is necessarily required today!)

DTFT Synthesis Equation	DTFT Analysis Equation
$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$	$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$
Parseval's Theorem	**N-point DFT**
$E = \sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$	$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn}, \quad W_N = e^{-j\frac{2\pi}{N}}$
Z-transform of a sequence $x[n]$	**Sinusoidal response of LTI systems, real h[n]**
$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$	$y[n] = \lvert H(e^{j\omega_0})\rvert \cos\left(\omega_0 n + \measuredangle H(e^{j\omega_0})\right)$

SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n - n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0\rvert R_x$
4	3.4.4	$nx[n]$	$-z\frac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

Additional Information (cont'd)

Geometric Sum	Geometric Series
$\sum_{k=0}^{N-1} q^k = \frac{1-q^N}{1-q}$	$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}, \quad \lvert q\rvert < 1$

Properties of the Discrete Fourier Transform

	Finite-Length Sequence (Length N)	N-point DFT (Length N)
1.	$x[n]$	$X[k]$
2.	$x_1[n], x_2[n]$	$X_1[k], X_2[k]$
3.	$ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
4.	$X[n]$	$Nx[((-k))_N]$
5.	$x[((n-m))_N]$	$W_N^{km} X[k]$
6.	$W_N^{-\ell n} x[n]$	$X[((k-\ell))_N]$
7.	$\sum_{m=0}^{N-1} x_1(m) x_2[((n-m))_N]$	$X_1[k]X_2[k]$
8.	$x_1[n]x_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1} X_1(\ell) X_2[((k-\ell))_N]$
9.	$x^*[n]$	$X^*[((-k))_N]$
10.	$x^*[((-n))_N]$	$X^*[k]$
11.	$\mathcal{R}e\{x[n]\}$	$X_{ep}[k] = \frac{1}{2}\{X[((k))_N] + X^*[((-k))_N]\}$
12.	$j\mathcal{I}m\{x[n]\}$	$X_{op}[k] = \frac{1}{2}\{X[((k))_N] - X^*[((-k))_N]\}$
13.	$x_{ep}[n] = \frac{1}{2}\{x[n] + x^*[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$
14.	$x_{op}[n] = \frac{1}{2}\{x[n] - x^*[((-n))_N]\}$	$j\mathcal{I}m\{X[k]\}$
	Properties 15–17 apply only when $x[n]$ is real.	
15.	Symmetry properties	$\begin{cases} X[k] = X^*[((-k))_N] \\ \mathcal{R}e\{X[k]\} = \mathcal{R}e\{X[((-k))_N]\} \\ \mathcal{I}m\{X[k]\} = -\mathcal{I}m\{X[((-k))_N]\} \\ \lvert X[k]\rvert = \lvert X[((-k))_N]\rvert \\ \sphericalangle\{X[k]\} = -\sphericalangle\{X[((-k))_N]\} \end{cases}$
16.	$x_{ep}[n] = \frac{1}{2}\{x[n] + x[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$
17.	$x_{op}[n] = \frac{1}{2}\{x[n] - x[((-n))_N]\}$	$j\mathcal{I}m\{X[k]\}$

SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
9. $[\cos \omega_0 n]u[n]$	$\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
10. $[\sin \omega_0 n]u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$
11. $[r^n \cos \omega_0 n]u[n]$	$\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\|z\| > r$
12. $[r^n \sin \omega_0 n]u[n]$	$\frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\|z\| > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$

Initial Value Theorem:

If $x[n]$ is a causal sequence, *i.e.* $x[n] = 0, \forall n < 0$, then

$$x[0] = \lim_{z\to\infty} X(z)$$