National Exams December 2014

07-Mec-B6, Advanced Fluid Mechanics

3 hours duration

Notes:

1. If doubt exists as to the interpretation of any question the candidate is urged to submit with the answer paper a clear statement of the assumptions made.
2. Candidates may use any non-communicating calculator. The exam is OPEN BOOK.
3. Answer any **4** of the 5 questions.
4. All questions are equally weighted.

Answer any **4** of the following 5 questions.

Question 1: A long municipal discharge pipe can be modeled as a source of strength m. The flow it induces can be modeled by the superposition of this source and a vortex of circulation, Γ, placed a distance a from the bottom of a deep lake as shown in Fig. Q1. Assuming that the waste water density is the same as the lake water, that the lake bed is flat, that the fluid velocity in the lake far from the discharge is negligible and that the free-surface effects can be neglected, determine:

a) The stream function that will represent this flow.
b) Verify that the lake bed is correctly simulated.
c) The velocity distribution along the lake bed.
d) The forces acting on the discharge pipe.

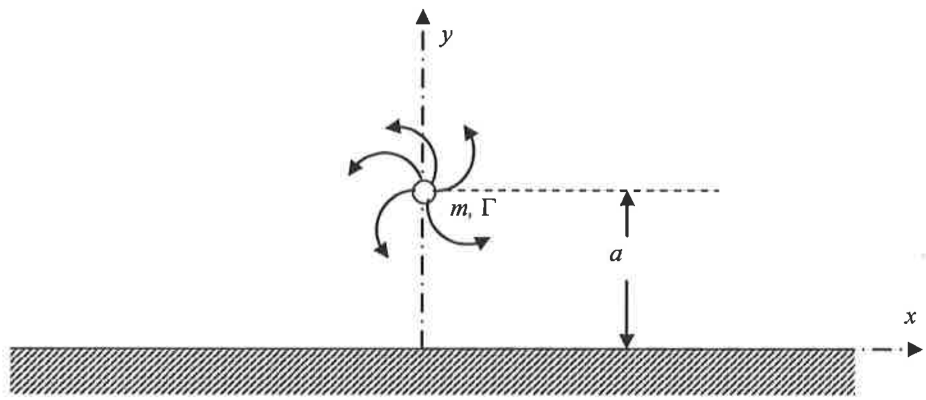


Figure Q1: Cross-sectional view of discharge pipe next to a lake bed.

Question 2: A pressurized air canister has an internal volume of 0.5 m^3. The air (γ = 1.4, R = 287 J/kg-k, Cp = 1004.5 J/kg-K) inside the canister is kept a constant temperature of 300K. Initially, the air inside the canister is pressurized to 10.0 MPa. The canister outlet consists of a valve shaped like a convergent-divergent nozzle. The throat area is given as $A_T = 1$ mm^2 and the exit area as $A_E = 3.5$ mm^2. The valve is opened to allow air to flow through. The air exhaust to atmosphere (P_b =100 kPa). It can be assumed that frictional losses are negligible. It can also be assumed that the air is still inside the canister (before entering the valve section).

a) Determine the flow rate, exit temperature and speed of the air immediately after opening the valve (i.e. when the internal pressure is 10.0 MPa).
b) Determine the flow rate, exit temperature and speed of the flow when the internal pressure of the canister reaches 5.0 MPa.
c) At what internal pressure does a shock form exactly at the exit of the convergent-divergent nozzle? What is the flow rate at this point? What is the air temperature and speed immediately after the exit?
d) Below what canister pressure is the valve (nozzle) no longer choked?

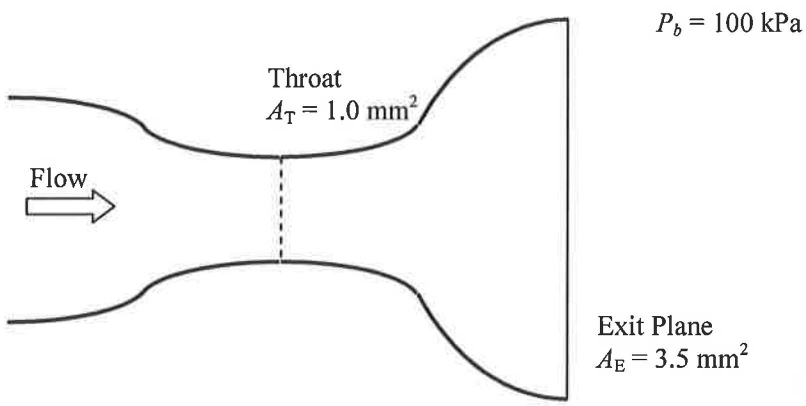


Figure Q2: Cross-sectional schematic of valve section.

Question 3: Water flows vertically and at a constant rate in a pipe of length L as shown in Fig. Q1. The pipe consists of a circular outer pipe of radius R and a central rod of radius $\delta = R/4$. The walls of the outer pipe are coated with Teflon so that the water does not wet the surface, so that the radial gradient of the streamwise velocity component is zero at the surface. The inner rod is not coated, so that water does wet the surface. The walls are non-porous. The inlet and outlet of the pipe are exposed to atmosphere, such that the inlet and outlet static pressures are the same. Assuming that $L/R >> 10$, that the flow is laminar and axisymmetric, determine:

a) The radial distribution of the radial velocity component u_r ;
b) The radial distribution of the vertical velocity component u_z ;
c) The force per unit length (magnitude and direction) acting on the inner rod.

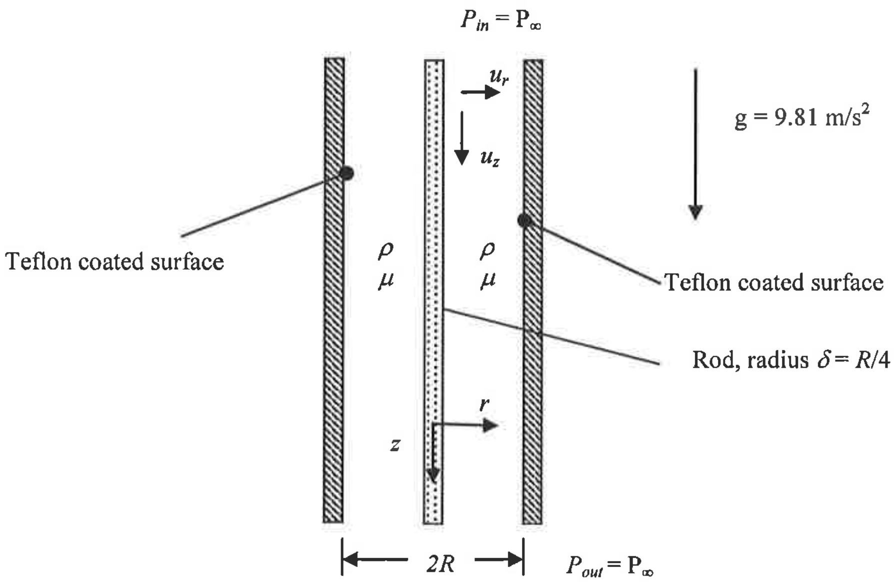


Figure Q3: Schematic of vertical pipe with rod.

Question 4: The flow in the inlet region of a two-dimensional channel can be modelled as two developing boundary layers on flat plates. Consider the case of water (of dynamic viscosity $\nu = \mu/\rho = 10^{-6}$ m^2/s) flowing into a channel of height $h = 2$ mm at an average (bulk) flow speed of $U_o = 5$ m/s. In this case, the Reynolds number based on h, $\mathrm{Re}_h = U_o h/\nu = 10{,}000$ and the flow will be turbulent.

The flow is said to be fully-developed downstream of the point where the two boundary layers meet (i.e. when $\delta = h/2$).

a) From the principle of conservation of mass, find the centre-line velocity, U_∞, as a function of the local boundary layer thickness δ.
b) If the inlet region is defined as extending from $x = 0$ to $x = L$, where at $x = L$ we require that $\delta = h/2$, estimate the inlet length in terms of the ratio L/h. What is the centre-line velocity at $x = L$?
c) What is the pressure gradient at $x = L$?
d) What is the wall shear stress, τ_w, at $x = L$?

In answering the question, assume that:

(i) the inlet velocity profile is uniform at U_o;
(ii) the flow in the boundary layers is turbulent;
(iii) that $\mathrm{Re}_x = U_\infty x/\nu$;
(iv) that the flow outside the boundary layers is irrotational;
(v) and:

$$\frac{\delta}{x} = \frac{0.2}{\mathrm{Re}_x^{1/7}}, \quad \frac{\delta^*}{\delta} = \frac{1}{8}, \quad \frac{\theta}{\delta} = \frac{7}{72}.$$

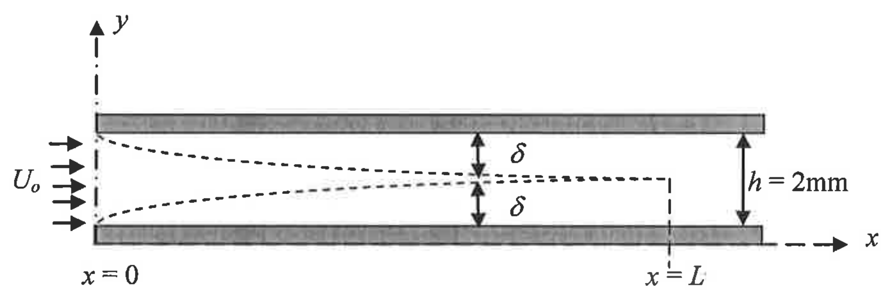


Figure Q4: Schematic of two-dimensional inlet region.

Question 5: You are working for a company which manufactures air compressors. A client would like to be able to handle air at P_{01}=101.3 kPa and T_{01}=250K at a rate of 200 m^3/s. The available company model has a rotor tip-to-tip diameter of 0.5m and handles atmospheric air at P_{01}=101.3 kPa and T_{01}=293K at a rate of 50 m^3/s when rotating at 5000 rpm. The total-to-total pressure ratio is 1.05. For a single stage you are given that the total-to-total efficiency is 0.93. The properties of air are: $\gamma = 1.4$; R = 287.0 J/kg-K and Cp = 1004.5 J/kg-K. Assuming that the stage efficiency does not change, determine: the required size of the geometrically similar compressor stage, its operating speed; the outlet total pressure and total temperature.

Aid Sheets

Compressible Flow:

Adiabatic flow: $$\frac{T_o}{T} = 1 + \frac{\gamma - 1}{2} M^2$$

Isentropic flow: $$\frac{P_o}{P} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}} \quad ; \quad \frac{\rho_o}{\rho} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{1}{\gamma-1}}$$

$$\dot{m} = \rho U A = \sqrt{\frac{\gamma}{R}} \cdot \frac{P_o}{\sqrt{T_o}} \cdot M \cdot \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}} A$$

Shock Relations: $$\frac{P_2}{P_1} = \frac{1 + \gamma M_1^{\ 2}}{1 + \gamma M_2^{\ 2}} = \frac{2\gamma M_1^{\ 2}}{\gamma + 1} - \frac{\gamma - 1}{\gamma + 1} \qquad M_2^2 = \frac{M_1^2 + \frac{2}{\gamma - 1}}{\frac{2\gamma}{\gamma - 1} M_1^2 - 1}$$

Boundary Layer Equations:

$$\frac{d}{dx}\left(U_o^2 \theta\right) + U_o \frac{dU_o}{dx} \cdot \delta^* = \tau_w / \rho \qquad \delta^* = \int_0^\delta \left(1 - \frac{u}{U_\infty}\right) dy \qquad \theta = \int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy$$

Laminar flow: $$C_{fx} = \frac{\tau_w(x)}{\frac{1}{2}\rho U_\infty^2} = \frac{0.67}{\mathrm{Re}_x^{1/2}}$$

Turbulent flow: $$C_{fx} = \frac{\tau_w(x)}{\frac{1}{2}\rho U_\infty^2} = \frac{0.0266}{\mathrm{Re}_x^{1/7}}$$

Conservation Equations for Cartesian Coordinate system

Continuity Equation:

$$\frac{D\rho}{Dt} + \rho\left(\nabla \bullet \vec{U}\right) = 0$$

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} \qquad \text{and} \qquad \nabla \bullet \vec{U} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

Linear Momentum:

***x*-direction:** $$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z} = -\frac{\partial P}{\partial x} + \rho g_x + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}$$

***y*-direction:** $$\rho\frac{\partial v}{\partial t} + \rho u\frac{\partial v}{\partial x} + \rho v\frac{\partial v}{\partial y} + \rho w\frac{\partial v}{\partial z} = -\frac{\partial P}{\partial y} + \rho g_y + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}$$

***z*-direction:** $$\rho\frac{\partial w}{\partial t} + \rho u\frac{\partial w}{\partial x} + \rho v\frac{\partial w}{\partial y} + \rho w\frac{\partial w}{\partial z} = -\frac{\partial P}{\partial z} + \rho g_z + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}$$

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\nabla \bullet \vec{U} \qquad \tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\tau_{yy} = 2\mu\frac{\partial v}{\partial y} - \frac{2}{3}\mu\nabla \bullet \vec{U} \qquad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \qquad \tau_{zz} = 2\mu\frac{\partial w}{\partial z} - \frac{2}{3}\mu\nabla \bullet \vec{U}$$

Conservation Equations for Cylindrical-polar Co-ordinate system

Continuity Equation: $$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r u_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho u_\theta)+\frac{\partial}{\partial z}(\rho u_z)=0$$

Linear Momentum Equations:

***r*-momentum:**

$$\rho\left[\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}+u_z\frac{\partial u_r}{\partial z}-\frac{u_\theta^2}{r}\right]$$
$$=-\frac{\partial P}{\partial r}+\rho g_r+\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr})+\frac{1}{r}\frac{\partial}{\partial \theta}(\tau_{r\theta})+\frac{\partial}{\partial z}(\tau_{rz})-\frac{\tau_{\theta\theta}}{r}$$

***θ*-momentum:**

$$\rho\left[\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+u_z\frac{\partial u_\theta}{\partial z}+\frac{u_r u_\theta}{r}\right]$$
$$=-\frac{1}{r}\frac{\partial P}{\partial \theta}+\rho g_\theta+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{\theta r})+\frac{1}{r}\frac{\partial}{\partial \theta}(\tau_{\theta\theta})+\frac{\partial}{\partial z}(\tau_{\theta z})$$

***z*-momentum:**

$$\rho\left[\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right]$$
$$=-\frac{\partial P}{\partial z}+\rho g_z+\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{zr})+\frac{1}{r}\frac{\partial}{\partial \theta}(\tau_{z\theta})+\frac{\partial}{\partial z}(\tau_{zz})$$

$$\tau_{rr}=\mu\left(2\frac{\partial u_r}{\partial r}-\frac{2}{3}\nabla\cdot\vec{U}\right) \qquad \tau_{\theta\theta}=\mu\left(2\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}\right)-\frac{2}{3}\nabla\cdot\vec{U}\right)$$

$$\tau_{zz}=\mu\left(2\frac{\partial u_z}{\partial z}-\frac{2}{3}\nabla\cdot\vec{U}\right) \qquad \tau_{r\theta}=\tau_{\theta r}=\mu\left(r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)$$

$$\tau_{rz}=\tau_{zr}=\mu\left(\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial z}\right) \qquad \tau_{\theta z}=\tau_{z\theta}=\mu\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta}+\frac{\partial u_\theta}{\partial z}\right)$$

$$\nabla\cdot\vec{U}=\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta)+\frac{\partial}{\partial z}(u_z)$$

$$\nabla\times\vec{U}=\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta}-\frac{\partial u_\theta}{\partial z}\right)\cdot\vec{e_r}+\left(\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right)\cdot\vec{e_\theta}+\left(\frac{1}{r}\frac{\partial (ru_\theta)}{\partial r}-\frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\cdot\vec{e_z}$$

Potential Flow

Stream functions:

Uniform Flow: $\Psi = U_o y = U_o r \sin\theta$

Source Flow: $\Psi = \dfrac{m}{2\pi}\tan^{-1}\left(\dfrac{y-y_o}{x-x_o}\right) = \dfrac{m}{2\pi}\theta$

Vortex Flow: $\Psi = -\dfrac{\Gamma}{4\pi}\ln\left[(x-x_o)^2+(y-y_o)^2\right] = -\dfrac{\Gamma}{2\pi}\ln r$

Doublet Flow: $\Psi = -\dfrac{\lambda(y-y_o)}{(x-x_o)^2+(y-y_o)^2} = -\lambda\dfrac{\sin(\theta)}{r}$

Potential functions:

Uniform Flow: $\Phi = U_o x = U_o r\cos\theta$

Source Flow: $\Phi = \dfrac{m}{4\pi}\ln\left[(x-x_o)^2+(y-y_o)^2\right] = \dfrac{m}{2\pi}\ln r$

Vortex Flow: $\Phi = \dfrac{\Gamma}{2\pi}\tan^{-1}\left(\dfrac{y-y_o}{x-x_o}\right) = \dfrac{\Gamma}{2\pi}\theta$

Doublet Flow: $\Phi = \dfrac{\lambda(x-x_o)}{(x-x_o)^2+(y-y_o)^2} = \lambda\dfrac{\cos(\theta)}{r}$

Velocity relationships:

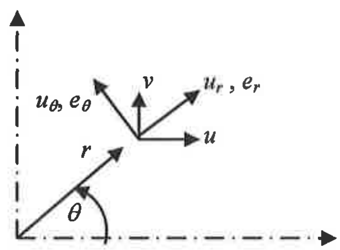
$$u = \frac{\partial\Phi}{\partial x} = \frac{\partial\Psi}{\partial y} \qquad v = \frac{\partial\Phi}{\partial y} = -\frac{\partial\Psi}{\partial x}$$

$$u_r = \frac{\partial\Phi}{\partial r} = \frac{1}{r}\frac{\partial\Psi}{\partial\theta} \qquad u_\theta = \frac{1}{r}\frac{\partial\Phi}{\partial\theta} = -\frac{\partial\Psi}{\partial r}$$

Transformation between Coordinate System

$r = \sqrt{x^2 + y^2}$ $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$

$x = r\cos\theta$ $y = r\sin\theta$



$\vec{e}_r = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$ $\vec{e}_\theta = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}$

$u_r = u\cos\theta + v\sin\theta$ $u_\theta = -u\sin\theta + v\cos\theta$

$u = u_r\cos\theta - u_\theta\sin\theta$ $v = u_r\sin\theta + u_\theta\cos\theta$