National Examination - 2016

Discrete Mathematics (04-BS-16)

Duration: 3 hours

Examination Type: Close Book, No aids allowed.

Instructions:

- This exam paper contains 13 pages (including this cover page).
- You have to answer 10 questions out of 12.
- Clearly indicate which questions you do not want to answer both on the cover page by crossing it and on the corresponding page by drawing a diagonal line across the page.

Question:	1	2	3	4	5	6	7	8	9	10	11	12	Total
Points:	10	10	10	10	10	10	10	10	10	10	10	10	100
Score:													

May 2016

1. Answer the following questions related to truth of propositions.

(a) 3 points Write the truth table for the compound proposition $p \leftrightarrow (\neg p \wedge q)$.
(Note: $\neg p$ is the negation of p.)

(b) 3 points Determine the truth value of "$\exists n \quad n+1 > n^2$" where the universe of discourse is all integers.

(c) 4 points Determine whether $\forall x(P(x) \rightarrow Q(x))$ has the same truth value as $\forall x\, P(x) \rightarrow \forall x\, Q(x)$.

2. Answer the following questions related to propositions and their relations.

(a) [4 points] Prove the following equivalence: $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$.

(b) [2 points] Determine the truth value of the following proposition "If $2 > 10$ then $\forall x \quad x = x + 1$."

(c) [2 points] Write the negation of the proposition "$\exists n \quad n + 1 > n^2$".

(d) [2 points] Find the dual of the proposition $(\neg p \wedge q) \vee \neg r$.

3. Answer the following questions related to set theory.

(a) **2 points** If $A = \{1,2,3\}$ and $B = \{1,3\}$, find $A \times B - B \times A$.

(b) **3 points** Let A, B and C be sets. Using algebra of sets show that $(A - B) - C^c = (A - C^c) - (B - C^c)$.

(c) **5 points** Let A, B and C be sets. Can we conclude $A = B$ if $A - C = B - C$ and $C - A = C - B$? If yes, prove it. If no, give a counter example.

4. Answer the following questions related to discrete probability.

(a) [5 points] In a class of 60 students 20 are girls and 40 are boys. In a recent test 16 of the girls have passed. Define the following two events. E1: A randomly chosen student from this class is a girl, E2: A randomly chosen student from this class has failed the course. Knowing that E1 and E2 are independent determine the number of boys that passed the test.

(b) [5 points] A lamp factory has three production lines A,B and C. Lines A,B and C produce 30%, 50% and 20% of the total respectively. Also, of the outputs of lines A, B and C, 4%, 8% and 3% are defective, respectively. A random lamp from this company is found the be defective. Find the probability that it was produced by line C.

5. Answer the following questions related to functions.

(a) [2 points] Determine if $f(n) = \sqrt{n^3 + n^2}$ is a function from $\mathbb{Z}$ to $\mathbb{R}$.

(b) [2 points] Determine if the function $f: \mathbb{R} \to \mathbb{R}$ and $f(x) = 3x^3 + 1$ is one-to-one.

(c) [2 points] Determine if the function $f: \mathbb{R} \to \mathbb{Z}$, and $f(x) = \lceil 1.5x \rceil$ is onto.

(d) [4 points] Consider functions f and g both defined from $\mathbb{R}$ to $\mathbb{R}$. Also $f(x) = x^2 + 1$ and $(f+g)(x) = x^3 + x^2$. Find $(g^{-1} \circ f)(x)$.

6. This is a question on counting.

Consider the permutations of the letters of the word **ASSIGNMENT**.

(a) [2 points] How many are there in total?

(b) [2 points] How many start with A and end with T?

(c) [2 points] How many start with a vowel?

(d) [2 points] How many have the block **SIGN**, with these four letters appearing in this order?

(e) [2 points] How many have the three vowels as a block?

7. Answer the following questions on relations.

(a) [4 points] Consider the relation R on set of all integers, where $(x,y) \in R$ if and only if $x = y \pm 3$. Determine if R is reflexive, symmetric, antisymmetric and/or transitive.

(b) [3 points] For R in part (a), is R^2 reflexive? Justify your answer.

(c) [3 points] How many relations can be defined on the set $A = \{1,2,3,4\}$.

8. This is a question on series and summations.

(a) 5 points Show that the sum of all elements of the set $A = \{2, 5, 8, \ldots 3n-1\}$ is $S = \frac{n(3n+1)}{2}$.

(b) 5 points Let us define $a_1 = 1$ and $a_i = a_{i-1} + 2i - 1$. Find a_n is closed form as a function of n.

9. This is a question on methods of proof.

(a) [5 points] Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$.

(b) [5 points] Find the flaw in the following "proof" that $a^n = 1$ for all nonnegative integers n, where a is a nonzero real number.

Basic Step: $a^0 = 1$ is true by definition of a^0 for $a \neq 0$.

Inductive Step: Assume $a^j = 1$ for all nonnegative integers j with $j \leq k$. Then note that

$$a^{k+1} = \frac{a^k \cdot a^k}{a^{k-1}} = \frac{1 \cdot 1}{1} = 1.$$

10. This is another question on methods of proof.

(a) [5 points] Show that at least six of any 36 days must fall on the same day of the week.

(b) [5 points] Show that for real numbers x and y, $|x| + |y| \geq |x + y|$.

11. This is a question on growth of functions and complexity of algorithms

(a) [4 points] Show that $f(x) = 3x^7 + x^5 \log x^9 + \frac{1}{x}$ is $\Omega(x^7)$.

(b) [6 points] Each part is two marks. The time complexity of Algorithms A and B are $\Theta(n^{10})$ and $\Theta(10^n)$ respectively. True or false.

- (T F) On a problem with size $n = 20$, it is certain that Algorithm B takes a longer time than A.
- (T F) Considering two problems with sizes n and $2n$, we expect Algorithm A to take about n^{10} times longer on the larger problem.
- (T F) There exist some n^* such that for any problem with size larger than n^* Algorithm B takes longer than A.

12. This is a question on graphs theory.

(a) [5 points] The adjacency matrix of graph G (with vertices a,b,c in the same order) is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

How many paths of length 3 exists between vertices b and c? How man between a and c?

(b) [5 points] Let $K_{m,n}$ denote the complete bipartite graph with m vertices on one side and n vertices on the other side. For what values of m and n does $K_{m,n}$ have an Euler path?