National Exams

04-CHEM-B1, Transport Phenomena

3 hours duration

NOTES

1. If doubt exists as to the interpretation of any question, the candidate is urged to submit with the answer paper, a clear statement of any assumptions made.
2. The examination is an OPEN BOOK EXAM.
3. Candidates may use any **non-communicating** calculator.
4. **Not all** problems are of equal weight.
5. **Answer all five questions.**
6. State all assumptions clearly.
7. The various conservation equations (continuity, momentum and shear stress-velocity gradient relationships, energy, and species conservation) are given in Tables 1-5 appended to this paper. These equations are also available in Brodkey, R.S. and Hershey H.C. (1988) *Transport Phenomena – A Unified Approach.*

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Q1. **[15 marks]** Making use of the continuity equation for an incompressible fluid flowing at steady state, show that a flow defined by the velocity field:

$$\vec{u} = (4t + 4x + 4y)\vec{i} + (t - 2y - 2z)\vec{j} + (t + x - 2z)\vec{k}$$

is possible in which $\vec{i}, \vec{j}$ and $\vec{k}$ are the unit vectors in the x, y, and z directions.

Q2. **[30 marks overall]** Consider the steady, low speed flow of a viscous isothermal fluid of density ρ and viscosity μ, between two infinitely long, parallel, vertical plates, spaced a distance h apart. The plate on the left (at $x = 0$) is stationary, whereas the plate on the right (at $x = h$) moves upward as constant speed, u_0.

(a) **[15 marks]** With reference to Fig. 1, and starting with the appropriate form of the Navier-Stokes equation, show that the velocity distribution is given by:

$$u_z = \frac{\rho g h^2}{2\mu}\left[\left(\frac{x}{h}\right)^2 - \left(\frac{x}{h}\right)\right] + u_0\left(\frac{x}{h}\right)$$

(b) **[5 marks]** Derive an expression for the velocity in the middle of the channel.

(c) **[10 marks]** Show that for the net mass flow rate to be zero, the right hand plate must move at a velocity given by:

$$u_0 = \frac{\rho g h^2}{6\mu}$$

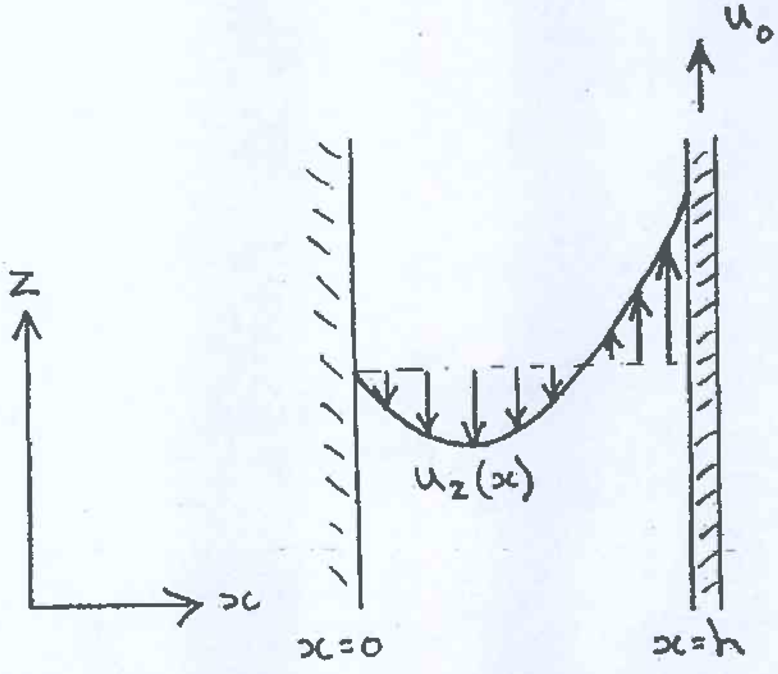

Fig. 1: Fully developed flow between vertical plates with one plate moving

Q3. **[25 marks overall]** Consider a hollow sphere of inner radius R_1 and outer radius R_2.

(i) [15 marks] If the inner surface is at the higher temperature T_1 and the outer surface is at the lower temperature T_2, starting with the appropriate form of the energy equation show that the temperature distribution throughout the shell is given by:

$$T = T_1 - \left\{ \frac{(T_1 - T_2) \cdot R_2 \cdot (r - R_1)}{r \cdot (R_2 - R_1)} \right\}$$

(ii) [10 marks] Furthermore, show that the heat loss at the outer surface of the sphere is given by:

$$\dot{Q} = -k \cdot 4\pi R_2^2 \cdot \frac{R_1 R_2 (T_1 - T_2)}{R_2^2 (R_1 - R_2)}$$

Q4. **[15 marks overall]** The instantaneous reaction $A \rightarrow 2B$ occurs on the outer surface of a long cylinder of length L with a diameter of 0.03 m. The ambient environment contains approximately 100 mol% A and is at 300 K and 1 atm.

(i) [10 marks] Show that an expression for the mole fraction profile is:

$$y_A = (1 + y_{As}) \left(\frac{r}{R} \right)^{-\left(\frac{\dot{N}_A}{2\pi L c D_{AB}} \right)} - 1$$

in which y_{As} is the mole fraction of A at the surface of the cylinder and $\dot{N}_A$ is the molar flow rate of A.

(ii) [5 marks] If $D_{AB} = 1.6 \times 10^{-5}$ m^2/s and the boundary layer thickness is 1 cm, determine the rate of formation of B per unit length of cylinder.

Q5. **[15 marks]** Oxygen is transferred from the lung cavity, across the lung tissue, to the network of blood vessels on the opposite side. The lung tissue may be viewed as a plane of thickness L. The inhalation process maintains a constant molar oxygen concentration, C_{Ai}, on the inner surface of the tissue. Due to absorption of oxygen by the blood, a constant concentration of C_{A0} is maintained on the other surface. Oxygen is consumed through metabolic processes in the lung tissue at a constant volumetric rate of $-k_0$. Show that an expression for the oxygen concentration profile in the lung tissue is:

$$C_A = C_{Ai} + \frac{k_0}{2D_{AB}} (z^2 - zL) + \frac{(C_{A0} - C_{Ai})}{L} z$$

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APPENDIX

Useful equations

Table 1: The Continuity Equation

$$\left[\partial\rho/\partial t+\left(\nabla\cdot\rho\vec{u}\right)\right]=0$$	(1)
Recatangular coordinates (*x*, *y*, *z*) $$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}\left(\rho u_x\right)+\frac{\partial}{\partial y}\left(\rho u_y\right)+\frac{\partial}{\partial z}\left(\rho u_z\right)=0$$	(1a)
Cylindrical coordinates (*r*, θ, *z*) $$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left(\rho r u_r\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\rho u_\theta\right)+\frac{\partial}{\partial z}\left(\rho u_z\right)=0$$	(1b)
Spherical coordinates (*r*, θ, ϕ) $$\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho r^2 u_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\rho u_\theta\sin\theta\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(\rho u_\varphi\right)=0$$	(1c)

Table 2: The Navier-Stokes equations for Newtonian fluids of constant ρ and μ

	$$\frac{\partial\vec{u}}{\partial t}+\left(\vec{u}\cdot\nabla\right)\vec{u}=-\frac{1}{\rho}\nabla P+\vec{g}+\nu\left(\nabla^2\vec{u}\right)$$	(2)
Rectangular coordinates (*x*, *y*, *z*)		
x-component	$$\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}+u_z\frac{\partial u_x}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+g_x+\nu\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right)$$	(2a)
y-component	$$\frac{\partial u_y}{\partial t}+u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}+u_z\frac{\partial u_y}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial y}+g_y+\nu\left(\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}+\frac{\partial^2 u_y}{\partial z^2}\right)$$	(2b)
z-component	$$\frac{\partial u_z}{\partial t}+u_x\frac{\partial u_z}{\partial x}+u_y\frac{\partial u_z}{\partial y}+u_z\frac{\partial u_z}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+g_z+\nu\left(\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_z}{\partial y^2}+\frac{\partial^2 u_z}{\partial z^2}\right)$$	(2c)

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Table 2: Continued

Cylindrical coordinates (*r*, θ, *z*)

r-component

$$\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}+u_z\frac{\partial u_r}{\partial z}-\frac{u_\theta^2}{r} =-\frac{1}{\rho}\frac{\partial P}{\partial r}+g_r+\nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(ru_r)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial^2 u_r}{\partial z^2}\right] \quad (2d)$$

θ-component

$$\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+u_z\frac{\partial u_\theta}{\partial z}+\frac{u_r u_\theta}{r} =-\frac{1}{\rho r}\frac{\partial P}{\partial \theta}+g_\theta+\nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(ru_\theta)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right] \quad (2e)$$

z-component

$$\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z} =-\frac{1}{\rho}\frac{\partial P}{\partial z}+g_z+\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial\theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right] \quad (2f)$$

Spherical coordinates (*r*, θ, *ϕ*)

r-component

$$\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}+\left(\frac{u_\varphi}{r\sin\theta}\right)\frac{\partial u_r}{\partial\varphi}-\frac{u_\theta^2}{r}-\frac{u_\varphi^2}{r}=-\frac{1}{\rho}\frac{\partial P}{\partial r}+g_r +\nu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}\left(r^2u_r\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_r}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_r}{\partial\varphi^2}\right] \quad (2g)$$

θ-component

$$\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\left(\frac{u_\varphi}{r\sin\theta}\right)\frac{\partial u_\theta}{\partial\varphi}+\frac{u_r u_\theta}{r}-\frac{u_\varphi^2}{r}\cot\theta=-\frac{1}{\rho r}\frac{\partial P}{\partial \theta}+g_\theta +\nu\left[\begin{array}{l}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(u_\theta\sin\theta\right)\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\theta}{\partial\varphi^2}\\ +\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}-\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}\end{array}\right] \quad (2h)$$

ϕ-component

$$\frac{\partial u_\varphi}{\partial t}+u_r\frac{\partial u_\varphi}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\varphi}{\partial \theta}+\frac{u_\varphi}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}+\frac{u_r u_\varphi}{r}+\frac{u_\theta u_\varphi}{r}\cot\theta=-\frac{1}{\rho r\sin\theta}\frac{\partial P}{\partial\varphi} +g_\varphi+\nu\left[\begin{array}{l}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\varphi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(u_\varphi\sin\theta\right)\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\varphi}{\partial\varphi^2}\\ +\frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\varphi}+\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial u_\theta}{\partial\varphi}\end{array}\right] \quad (2i)$$

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Table 3: Shear stress-velocity gradient relationships for a constant viscosity Newtonian fluid

$$[\hat{\tau}] = -\mu\left[\nabla\hat{u} + (\nabla\hat{u})^T\right] + \frac{2}{3}\nabla\cdot\vec{u} \tag{3}$$

Cartesian coordinates (*x*, *y*, *z*)

$$\tau_{xx} = -\mu\left[2\frac{\partial u_x}{\partial x}\right] + \frac{2}{3}\nabla\cdot\vec{u} \tag{3a}$$

$$\tau_{yy} = -\mu\left[2\frac{\partial u_y}{\partial y}\right] + \frac{2}{3}\nabla\cdot\vec{u} \tag{3b}$$

$$\tau_{zz} = -\mu\left[2\frac{\partial u_z}{\partial z}\right] + \frac{2}{3}\nabla\cdot\vec{u} \tag{3c}$$

$$\tau_{xy} = -\mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = \tau_{yx} \tag{3d}$$

$$\tau_{yz} = -\mu\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) = \tau_{zy} \tag{3e}$$

$$\tau_{zx} = -\mu\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) = \tau_{xz} \tag{3f}$$

Cylindrical coordinates (*r*, θ, *z*)

$$\tau_{rr} = -\mu\left[2\frac{\partial u_r}{\partial r}\right] + \frac{2}{3}\nabla\cdot\vec{u} \tag{3g}$$

$$\tau_{\theta\theta} = -2\mu\left[\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right] + \frac{2}{3}\mu\nabla\cdot\vec{u} \tag{3h}$$

$$\tau_{zz} = -\mu\left[2\frac{\partial u_z}{\partial z}\right] + \frac{2}{3}\mu\nabla\cdot\vec{u} \tag{3i}$$

$$\tau_{r\theta} = -\mu\left[r\frac{\partial}{\partial r}(u_\theta/r) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right] = \tau_{\theta r} \tag{3j}$$

$$\tau_{\theta z} = -\mu\left[\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right] = \tau_{z\theta} \tag{3k}$$

$$\tau_{\theta z} = -\mu\left[\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right] = \tau_{z\theta} \tag{3l}$$

Spherical coordinates (r, θ, ϕ)

$$\tau_{rr} = -\mu\left[2\frac{\partial u_r}{\partial r}\right] + \frac{2}{3}\nabla\cdot\vec{u} \quad (3m)$$

$$\tau_{\theta\theta} = -2\mu\left[\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right] + \frac{2}{3}\mu\nabla\cdot\vec{u} \quad (3n)$$

$$\tau_{\phi\phi} = -2\mu\left[\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + (u_\theta/r)\cot\theta\right] + \frac{2}{3}\mu\nabla\cdot\vec{u} \quad (3o)$$

$$\tau_{r\theta} = -\mu\left[r\frac{\partial}{\partial r}(u_\theta/r) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right] = \tau_{\theta r} \quad (3p)$$

$$\tau_{\theta\phi} = -\mu\left[\frac{\sin\theta}{r}\left[\frac{\partial}{\partial \theta}\left(\frac{u_\phi}{\sin\theta}\right)\right] + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}\right] = \tau_{\phi\theta} \quad (3q)$$

$$\tau_{r\phi} = -\mu\left[\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} + r\frac{\partial}{\partial r}(u_\phi/r)\right] = \tau_{\phi r} \quad (3r)$$

Table 4: The Energy Equation for Incompressible media

$$\frac{\partial(\rho c_P T)}{\partial t} + (\vec{u}\cdot\nabla)(\rho c_P T) = [\nabla\cdot\alpha\nabla(\rho c_P T)] + \dot{T}_G \quad (4)$$

Rectangular coordinates (x, y, z)

$$\frac{\partial T}{\partial t} + u_x\frac{\partial T}{\partial x} + u_y\frac{\partial T}{\partial y} + u_z\frac{\partial T}{\partial z} = \frac{\partial}{\partial x}\left(\alpha\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\alpha\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\alpha\frac{\partial T}{\partial z}\right) + \frac{\dot{T}_G}{\rho c_P} \quad (4a)$$

Cylindrical coordinates (r, θ, z)

$$\frac{\partial T}{\partial t} + u_r\frac{\partial T}{\partial r} + \frac{u_\theta}{r}\frac{\partial T}{\partial \theta} + u_z\frac{\partial T}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\alpha\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\alpha\frac{\partial T}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(\alpha\frac{\partial T}{\partial z}\right) + \frac{\dot{T}_G}{\rho c_P} \quad (4b)$$

Spherical coordinates (r, θ, ϕ)

$$\frac{\partial T}{\partial t} + u_r\frac{\partial T}{\partial r} + \frac{u_\theta}{r}\frac{\partial T}{\partial \theta} + \frac{u_\varphi}{r\sin\theta}\frac{\partial T}{\partial \varphi} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\alpha\frac{\partial T}{\partial r}\right)$$
$$+ \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\alpha\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \varphi}\left(\alpha\frac{\partial T}{\partial \varphi}\right) + \frac{\dot{T}_G}{\rho c_P} \quad (4c)$$

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Table 5: The continuity equation for species *A*

$$\frac{\partial C_A}{\partial t} + (\vec{u}\cdot\nabla)C_A = D_A \nabla^2 C_A + \dot{R}_{A,G} \tag{5}$$

Rectangular coordinates (*x*, *y*, *z*)

$$\frac{\partial C_A}{\partial t} + u_x \frac{\partial C_A}{\partial x} + u_y \frac{\partial C_A}{\partial y} + u_z \frac{\partial C_A}{\partial z} = \frac{\partial}{\partial x}\left(D\frac{\partial C_A}{\partial x}\right) + \frac{\partial}{\partial y}\left(D\frac{\partial C_A}{\partial y}\right) + \frac{\partial}{\partial z}\left(D\frac{\partial C_A}{\partial z}\right) + \dot{R}_{A,G} \tag{5a}$$

Incompressible media, Cylindrical coordinates (*r*, θ, *z*)

$$\frac{\partial C_A}{\partial t} + u_r \frac{\partial C_A}{\partial r} + \frac{u_\theta}{r}\frac{\partial C_A}{\partial \theta} + u_z \frac{\partial C_A}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(rD\frac{\partial C_A}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(D\frac{\partial C_A}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(D\frac{\partial C_A}{\partial z}\right) + \dot{R}_{A,G} \tag{5b}$$

Incompressible media, Spherical coordinates (*r*, θ, ϕ)

$$\frac{\partial C_A}{\partial t} + u_r \frac{\partial C_A}{\partial r} + \frac{u_\theta}{r}\frac{\partial C_A}{\partial \theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial C_A}{\partial \phi} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 D\frac{\partial C_A}{\partial r}\right)$$
$$+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(D\sin\theta\frac{\partial C_A}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(D\frac{\partial C_A}{\partial \phi}\right) + \dot{R}_{A,G} \tag{5c}$$